Kvazinormalni modovi crnih rupa

photo: Event Horizon Telescope

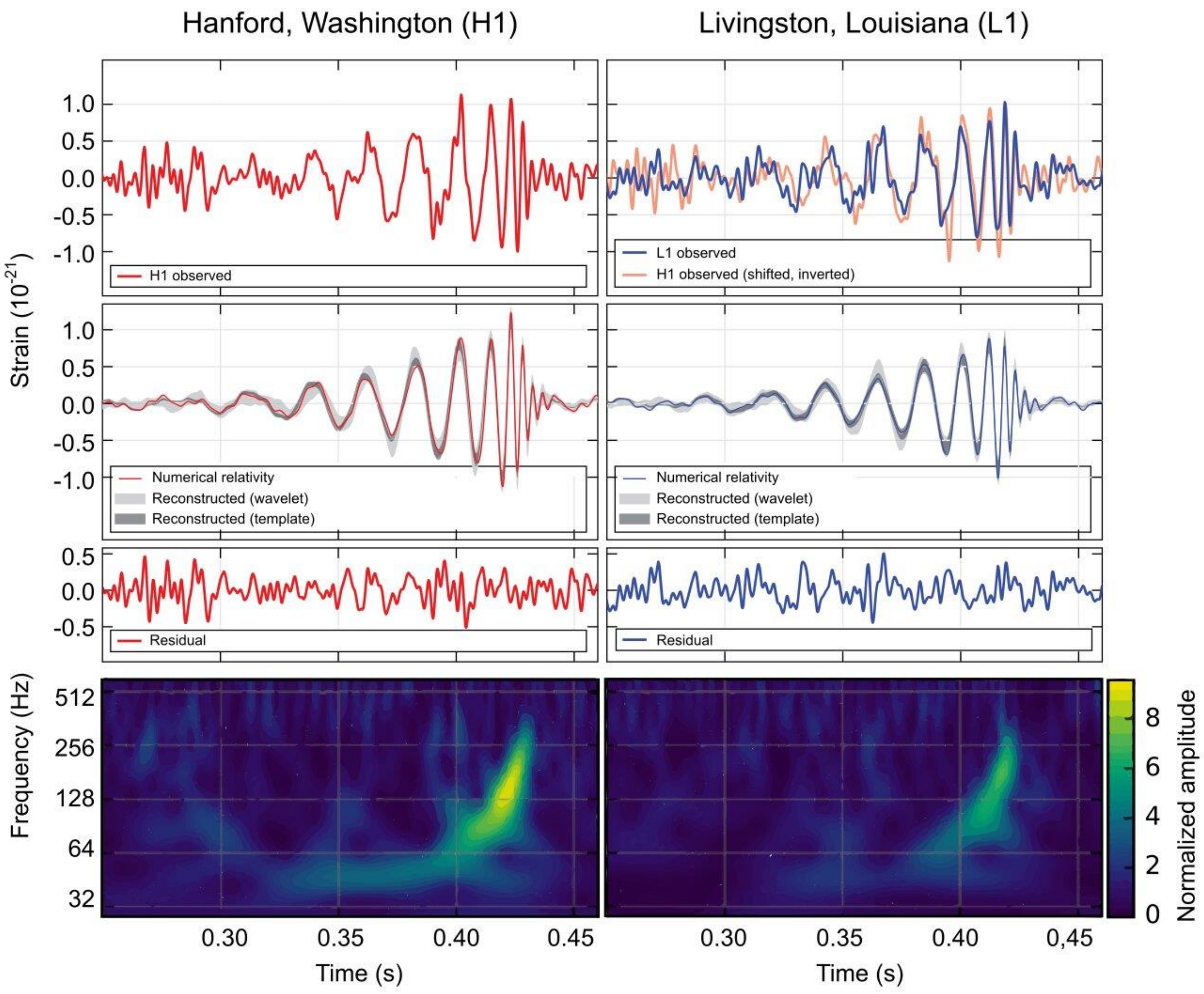
Lovro Dulibić

mentor: izv. prof. Ivica Smolić

Ukratko

- crne rupe su **disipativni sustavi** – gravitacijskim valovima odvode energiju

- **kvazinormalni modovi** su svojstveni modovi disipativnih sustava



LIGO Scientific Collaboration and Virgo Collaboration

Perturbacije crnih rupa

	Nerotirajuća	**Rotirajuća**
Nenabijena	Schwarzchild	Kerr
Nabijena	Reissner-Nordström	Kerr-Newman

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- tenzorske perturbacije $(j = 2)$ - metrika
- vektorske perturbacije $(j = 1)$ - elektromagnetsko polje
- skalarne perturbacije $(j = 0)$ - najjednostavniji primjer

Klein-Gordon + Schwarzschild
(skica izvoda)

$$\Box\phi = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\partial^\mu)\phi = 0$$

...

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(\sin^2\theta d\varphi^2 + d\theta^2)$$

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- supstitucija (*eng. tortoise coordinate*) – **kornjačasta koordinata**

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$$\frac{dr_*}{dr} = \left(1 - \frac{2M}{r}\right)^{-1} \qquad r_* = r + 2M\ln\left(\frac{r}{2M} - 1\right)$$

horizont $r = 2M \ \rightarrow r_* = -\infty$

beskonačnost $r = +\infty \rightarrow r_* = +\infty$

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Klein-Gordon + Schwarzschild
(skica izvoda)

- na kraju imamo ODJ drugog reda, nalik Schrödingerovoj

$$\frac{d^2\psi}{dr_*^2} + \left(\omega^2 + V(r_*)\right)\psi = 0$$

$$V(r) = \left(1 - \frac{2M}{r}\right)\left(\frac{l(l+1)}{r^2} + \frac{2M}{r^3}(1 - j^2)\right)$$

$V(r) \to 0$ kada $r \to 2M$ ili $r \to +\infty$

Definicija kvazinormalnih modova

- horizont – isključivo **ulazeći** valovi
- beskonačnost – isključivo **izlazeći** valovi

- **trnu** s vremenom, dakle $\omega \in \mathbb{C}$, $\mathrm{Im}(\omega) > 0$

uz vremensku ovisnost $e^{+i\omega t}$

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$\psi \propto e^{+i\omega r_*}$, kada $r_* \to -\infty$ **horizont**

$\psi \propto e^{-i\omega r_*}$, kada $r_* \to +\infty$ **beskonačnost**

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Definicija kvazinormalnih modova

- divergirajuća rješenja?

$$\psi \propto e^{i\omega t} e^{+i\omega r_*} = e^{i\omega_R (t+r_*) - \omega_I (t+\underline{\underline{r_*}})} \qquad r_* \to -\infty$$
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$$\psi \propto \left(\frac{r}{2M} - 1\right)^{+2M\omega i} e^{+i\omega r} \qquad r \to 2M$$

$$\psi \propto \left(\frac{r}{2M} - 1\right)^{-2M\omega i} e^{\underline{\underline{-i\omega r}}} \qquad r \to +\infty$$

Problem je postavljen

$$\frac{d^2\psi}{dr_*^2} + \left(\omega^2 + V(r_*)\right)\psi = 0$$

$\psi \propto e^{+i\omega r_*}$, kada $r_* \to -\infty$ **horizont**

$\psi \propto e^{-i\omega r_*}$, kada $r_* \to +\infty$ **beskonačnost**

$$V(r) = \left(1 - \frac{2M}{r}\right)\left(\frac{l(l+1)}{r^2} + \frac{2M}{r^3}(1 - j^2)\right)$$

Ideja i plan

- cilj je pronaći **analitički** izraz za **frekvenciju** kvazinormalnih modova

- **kako?**
    1. analitički produljiti ODJ na kompleksnu ravninu

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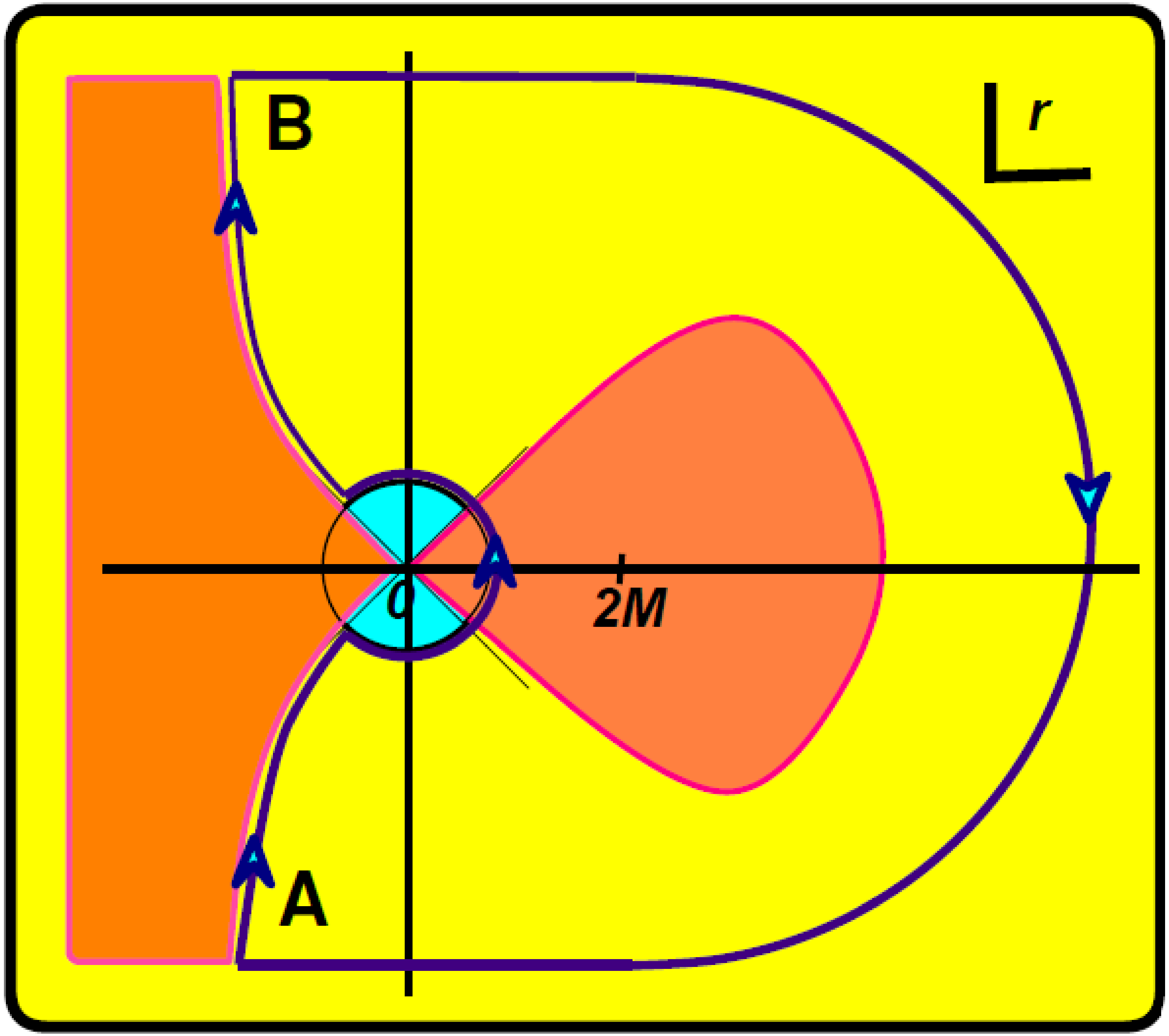
  1. analitički produljiti ODJ na kompleksnu ravninu
  2. rubni uvjeti odredit će **monodromiju** rješenja
  3. izračunati monodromiju ‘na prste’ i zahtijevati da poštuje rubni uvjet

$$r_* = r + 2M \ln\left(\frac{r}{2M} - 1\right)$$

- tamno područje $\mathrm{Re}(r_*) < 0$

L. Motl, A. Neitzke, *Asymptotic black hole quasinormal frequencies*, Adv. Theor. Math. Phys. 7 (2003) 307–330



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$$r_{*I} = -\frac{\omega_I}{\omega_R} r_{*R}, \text{ kada } \omega_I \gg \omega_R, \text{ tada } r_{*R} \to 0$$

$$r_* = r + 2M \ln\left(\frac{r}{2M} - 1\right)$$

- tamno područje $\mathrm{Re}(r_*) < 0$

$$r_*(r = 0) = 2M\pi i$$

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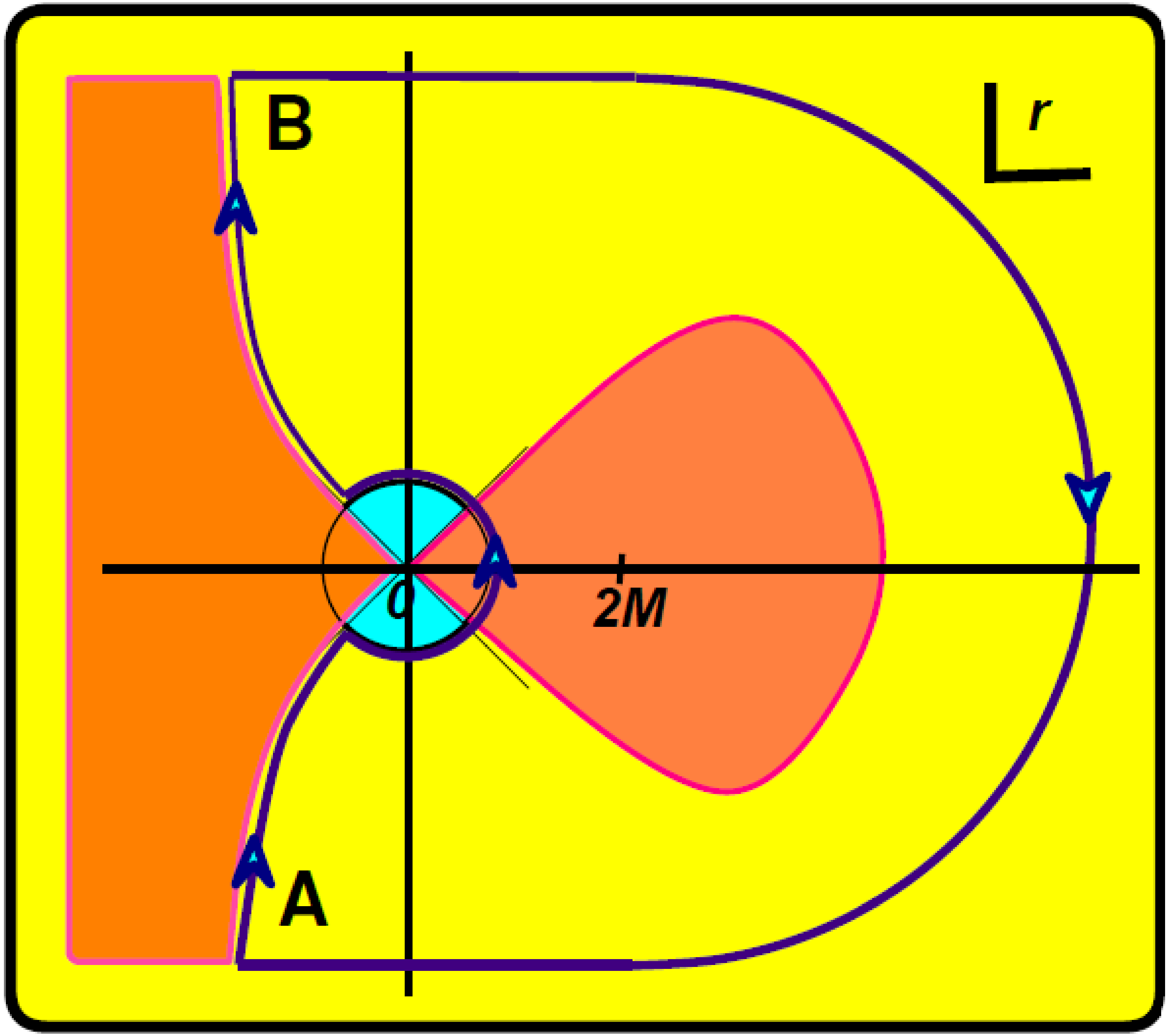

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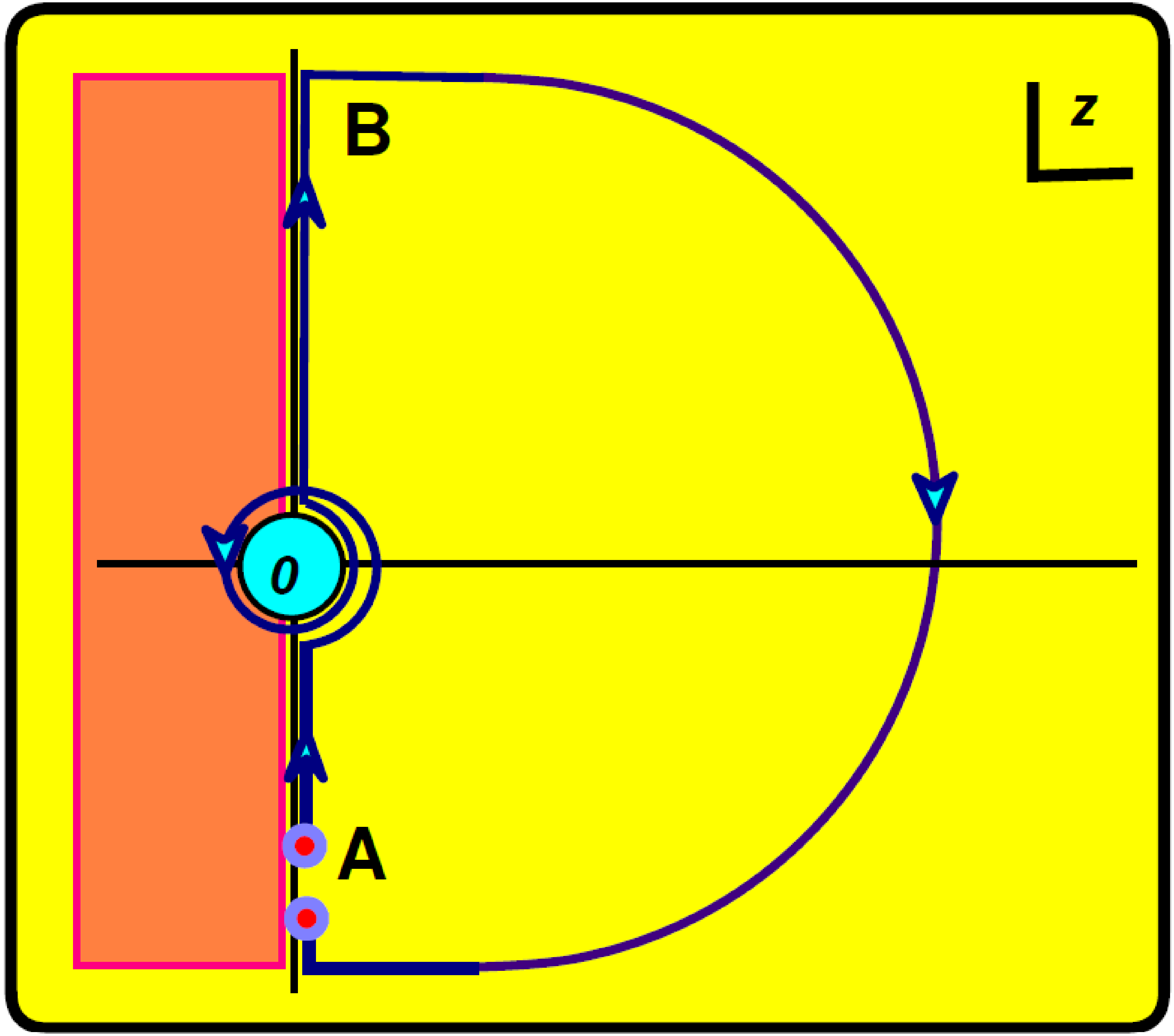

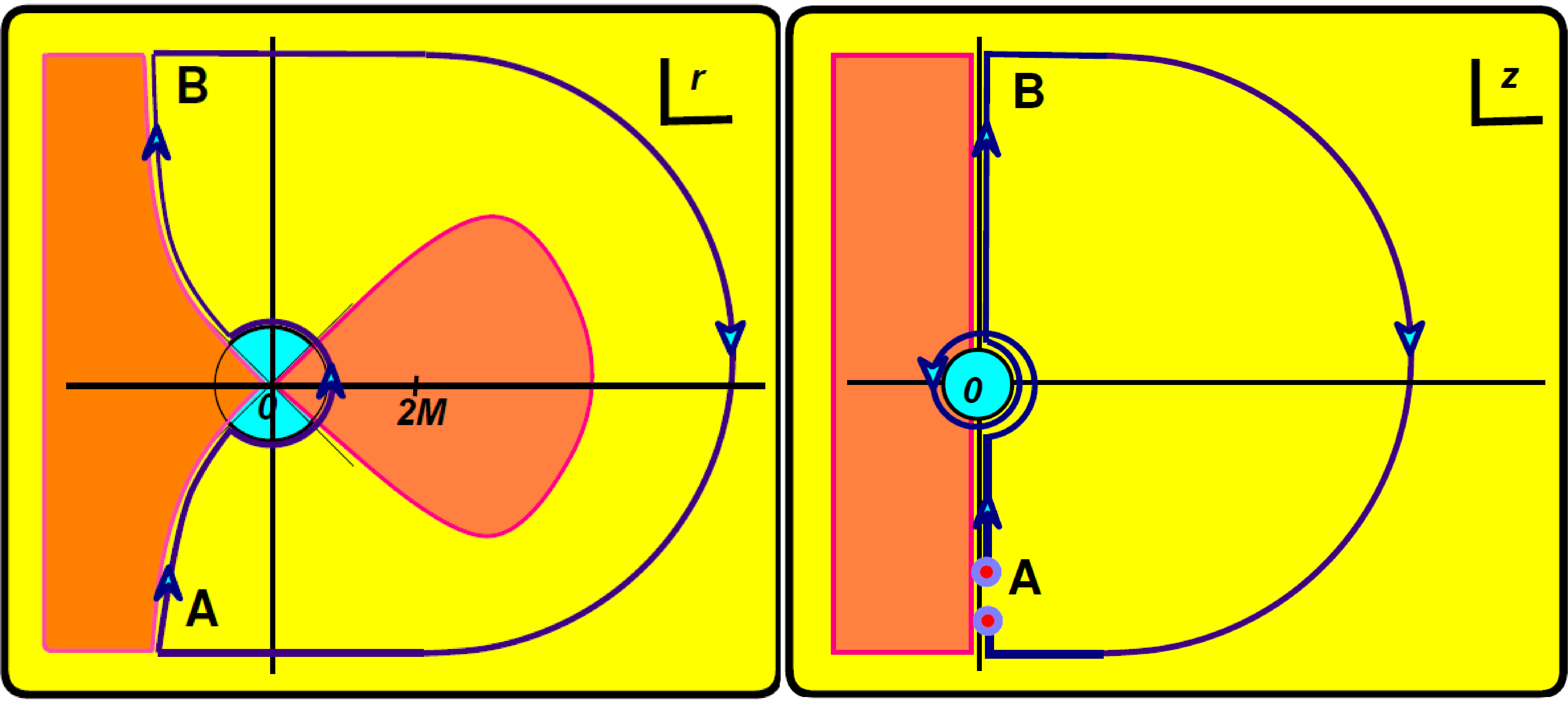

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Razvoj oko $z = 0$

$$z = -2Mi\pi + r + 2M\ln\left(\frac{r}{2M} - 1\right)$$

$$\frac{d^2\psi}{dz^2} + \left(\omega^2 + V(z)\right)\psi = 0$$

Razvoj oko $z = 0$

$$z \approx -2Mi\pi + r + 2M\left(i\pi - \frac{r}{2M} - \frac{r^2}{8M^2} - \cdots\right) = -\frac{r^2}{4M}$$

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$$\psi \propto A_+ c_+ \sqrt{\omega z} J_{+\frac{j}{2}}(\omega z) + A_- c_- \sqrt{\omega z} J_{-\frac{j}{2}}(\omega z)$$

A područje

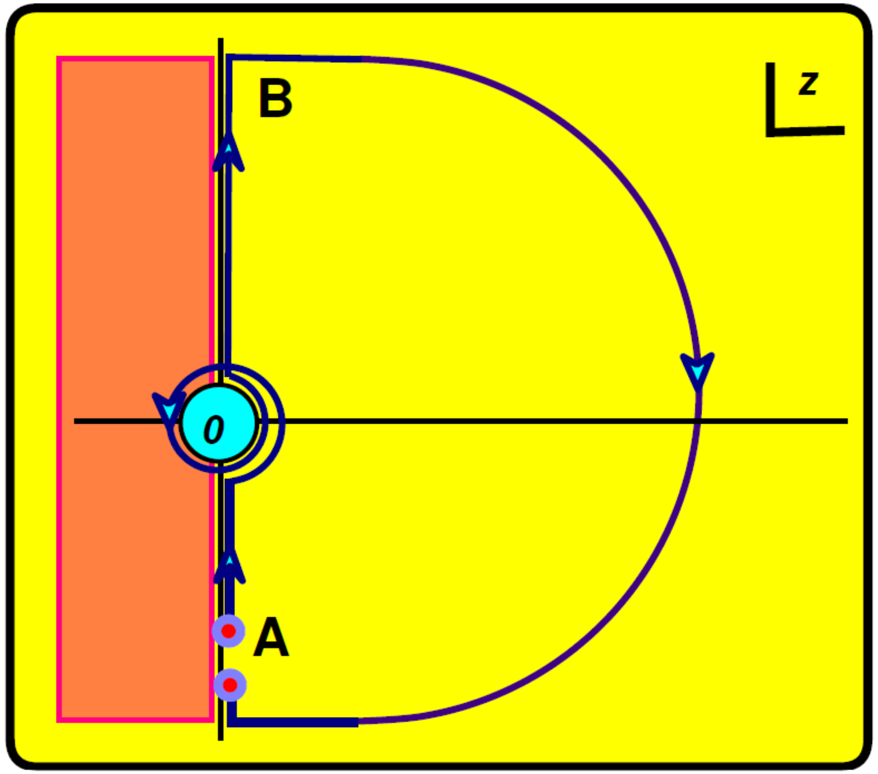
- iz nule razvijamo rješenje po pravcu

$$\mathrm{Im}(\omega z) = 0, \omega z \to +\infty$$

$$c_\pm J_{\pm\frac{j}{2}}(\omega z) \to 2\cos(\omega z - \alpha_\pm)$$

$$\alpha_\pm = \frac{\pi}{4}(1 \pm j)$$

$$\psi_A \propto \left(A_+ e^{i\alpha_+} + A_- e^{i\alpha_-}\right) e^{-i\omega z}$$



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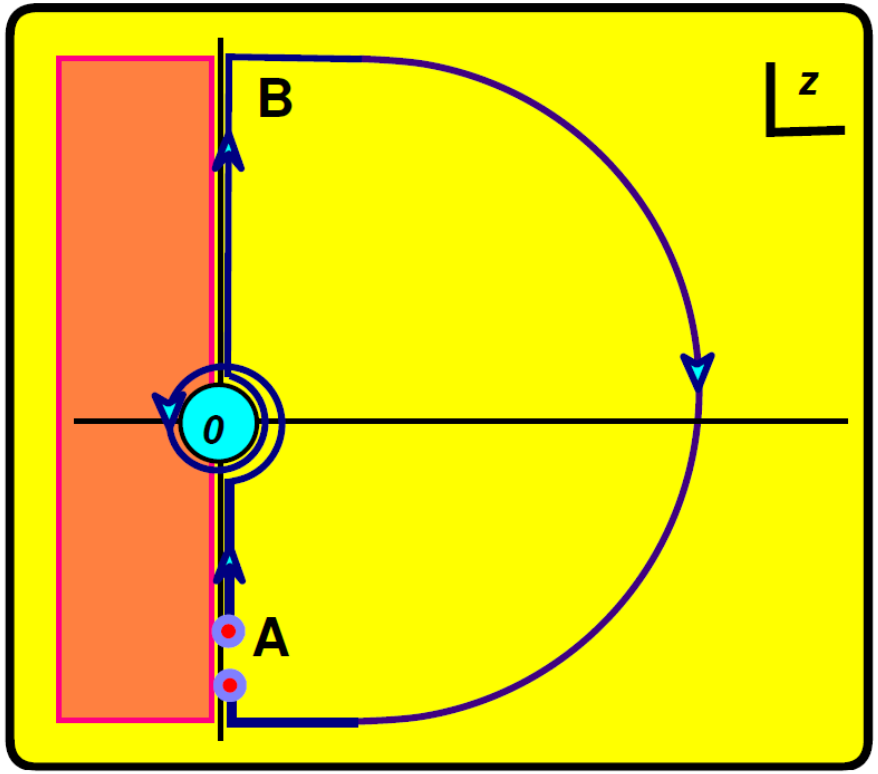
3π rotacija oko $z = 0$

$$\psi \propto A_+ c_+ \sqrt{\omega z} J_{+\frac{j}{2}}(\omega z) + A_- c_- \sqrt{\omega z} J_{-\frac{j}{2}}(\omega z)$$

- iz svojstava rješenja blizu nule
  znamo da ta rotacija daje faktor $e^{6i\alpha_\pm}$



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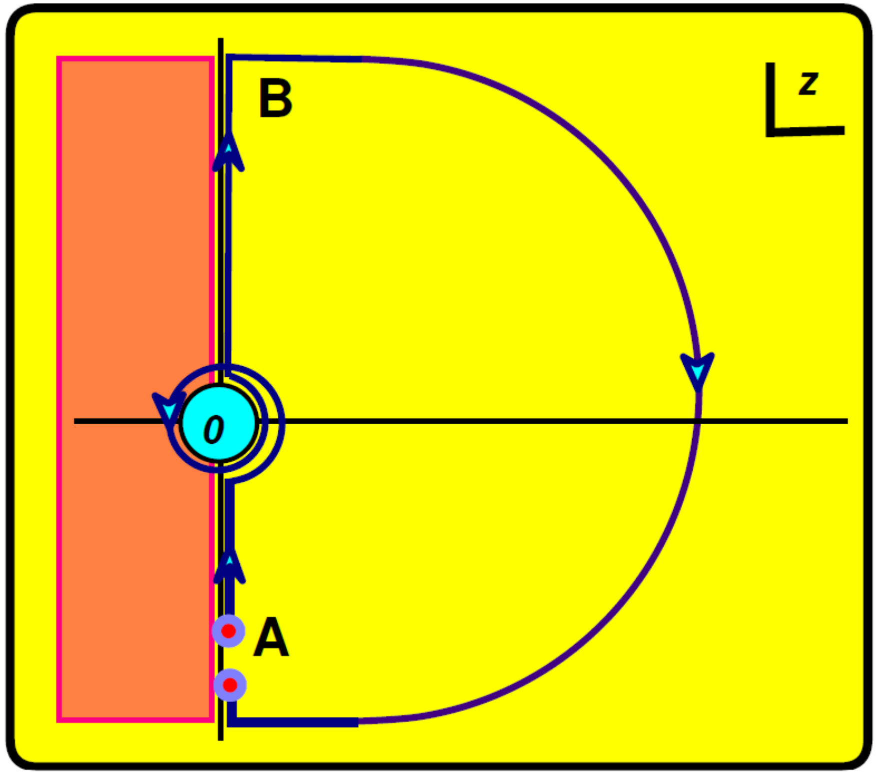
B područje

- iz nule razvijamo rješenje po pravcu

$$\mathrm{Im}(\omega z) = 0, \omega z \to -\infty$$

$$c_{\pm} J_{\pm\frac{j}{2}}(\omega z) \to e^{6i\alpha_{\pm}} 2\cos(-\omega z - \alpha_{\pm})$$



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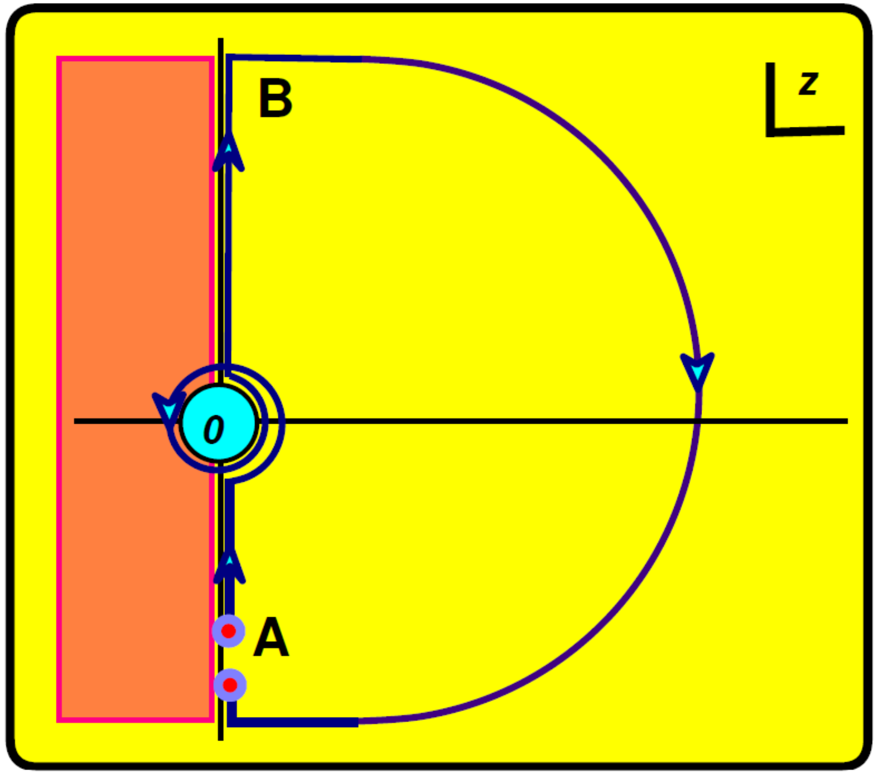
$$\psi_B \propto \left(A_+ e^{5i\alpha_+} + A_- e^{5i\alpha_-}\right) e^{-i\omega z} + \left(A_+ e^{7i\alpha_+} + A_- e^{7i\alpha_-}\right) e^{+i\omega z}$$

Duž polukruga nazad do A

1. A područje $\left(A_+ e^{i\alpha_+} + A_- e^{i\alpha_-}\right)$
2. B područje $\left(A_+ e^{5i\alpha_+} + A_- e^{5i\alpha_-}\right)$
3. monodromija **izlazećih** $e^{-2\pi\omega 2M}$



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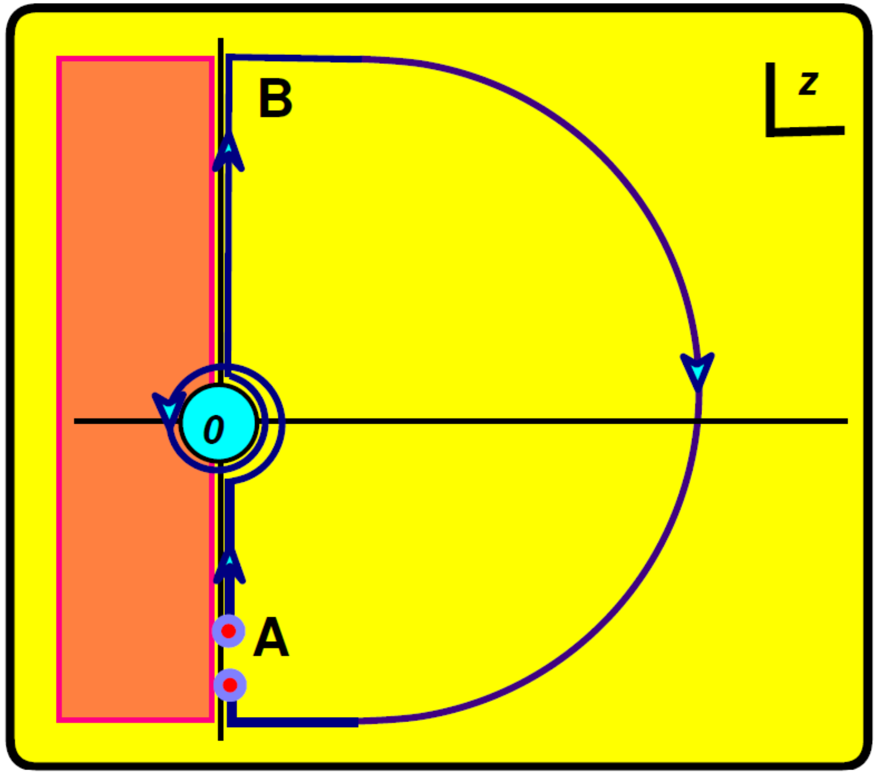

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$$\frac{\left(A_+ e^{5i\alpha_+} + A_- e^{5i\alpha_-}\right) e^{-2\pi\omega 2M}}{\left(A_+ e^{i\alpha_+} + A_- e^{i\alpha_-}\right)} = -e^{-2\pi\omega 2M}\left(1 + 2\cos(\pi j)\right)$$

Konačno, frekvencija

- rubni uvjet da na horizontu moraju biti isključivo ulazeći valovi implicira da monodromija **rješenja** mora biti $e^{2\pi\omega 2M}$

- izjednačavanjem izračunatog i zahtijevanog

$$e^{2\pi\omega 2M} = -e^{-2\pi\omega 2M}(1 + 2\cos(\pi j))$$

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$$4\pi\omega 2M = (2n + 1)i\pi \pm \ln(1 + 2\cos(\pi j))$$

Zaključno

- crne rupe su **disipativni sustavi** koji gube energiju gravitacijskim valovima
- kvazinormalni modovi **ne ovise o izvoru perturbacije**, već o svojstvima same crne rupe

- **produljenjem na kompleksnu ravninu** rubni uvjet preveli smo u zahtjev na monodromiju
- rješavanjem jednadžbe u **asimptotskim područjima** u kojima znamo rješenje izračunali smo monodromiju te dobili izraz za **frekvencije** kvazinormalnih modova

Hvala na pažnji!

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mentor: izv. prof. Ivica Smolić