Kvantne korekcije entropije crne rupe

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Uvod

- Crne rupe: dijelovi prostorvremena iz kojeg ni objekti ni svjetlost, ne mogu izaći, neovisno o brzini kojom se gibaju

$$R = \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$ds^2 = -\left(1 - \frac{2MG}{r}\right) dt^2 + \left(1 - \frac{2MG}{r}\right)^{-1} dr^2 + r^2 d\Omega$$

- Stacionarne crne rupe su opisane s 3 parametra: M, Q, J

- Motivacija za uvođenje termodinamičkih zakona: upad materije u crnu rupu

Zakoni mehanike crnih rupa

- 0. Za stacionarnu crnu rupu vrijedi $\kappa = cost.$
- 1. $dM = \frac{\kappa}{8\pi} dA + \Omega dJ$
- 2. $\frac{dA}{dt} \geq 0$
- 3. Nema fizikalnog procesa koji u konačno mnogo operacija može κ smanjiti na 0

$$dM = \frac{\kappa}{8\pi} dA + \Omega dJ$$

$$dE = TdS - pdV$$

$$T_H = \frac{\hbar\kappa}{2\pi}$$

$$S_{BH} = \frac{A}{4\ell_{Pl}^2}$$

$$\ell_{Pl}^2 = G\hbar$$

- Schwarzschildova crna rupa: $S_{BH} = \frac{4\pi}{\hbar} G M^2$

- Važnost entropije: uvid u mikroskopsku strukturu gravitacije $S = k_B \ln(\Omega)$

- Broj energijskih nivoa dostupnih čestici blizu horizonta crne rupe divergira

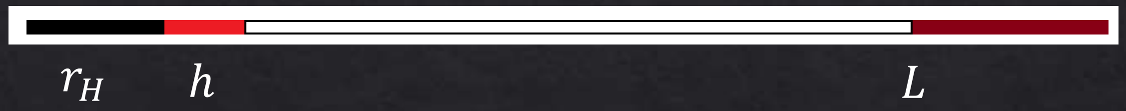
- Kvantna teorija polja: UV divergencija na horizontu → potrebno regularizirati

Brick wall metoda za Schwarzschildovu crnu rupu

- Razmatramo kvantna polja koja propagiraju unutar fiksne pozadine crne rupe

- Određujemo broj mogućih mikrostanja brick wall metodom

$$\Phi(r) = 0, \qquad r \leq r_H + h, \qquad r \geq L$$



$$\Box\Phi = 0 \qquad \Box = \frac{1}{\sqrt{-g}}\partial_\mu\left(g^{\mu\nu}\sqrt{-g}\partial_\nu\right)$$

$$\Phi = e^{-iEt/\hbar}R_{\ell m}(r)Y_{\ell m}(\Omega)$$

$$f(r)R'' + \frac{2}{r}f(r)R' - \left[\frac{\ell(\ell+1)}{r^2} - \frac{E^2}{f(r)}\right]R = 0$$

$$f(r) = 1 - \frac{2GM}{r}$$

- WKB approksimacija $R_{\ell m}(r) = \exp\left[\frac{i}{\hbar}\int^{r} dr' k(r')\right]$

$$k^2 = \frac{1}{f(r)}\left[\frac{E^2}{f(r)} - \frac{\hbar^2 \ell(\ell+1)}{r^2}\right]$$

$$k^2 \geq 0$$

$$\ell_{max}(\ell_{max}+1) = \frac{E^2 r^2}{\hbar^2 f(r)}$$

- Bohr-Sommerfeldovo pravilo za kvantizaciju $\pi n_r = \int_{r_H+h}^{L} k(r,\ell,E,m)dr$

$$N(E) = \sum_{\ell=0}^{\ell_{max}} \sum_{m=-l}^{l} n_r \longrightarrow N(E) = \frac{1}{\pi}\int_{2GM+h}^{L} dr \int_0^{\ell_{max}} d\ell(2\ell+1)k(r,\ell,E)$$

$$x = r - 2GM \qquad \int_h^{L-2GM} dx = \underbrace{\int_h^R dx}+ \underbrace{\int_R^{L-2GM} dx}$$

Dominiraju članovi koji divergiraju za $h \to 0$ $L \gg 2GM$

$$N(E) = \frac{32G^4M^4E^3}{3\pi h} + \frac{E^3L^3}{\pi}$$

Doprinos vakuuma

$$F = -\int dE \frac{N(E)}{e^{\beta E}+1} \longrightarrow F = -\frac{2\pi^3}{45h}\left(\frac{2GM}{\beta}\right)^4 - \frac{L^3\pi^3}{15\beta^4}$$

$$S = \beta^2 \frac{\partial F}{\partial \beta} \longrightarrow S = \frac{8\pi^3}{45h}\frac{(2GM)^4}{\beta^3}$$

$$S_{BH}, \beta = \frac{1}{T_H}, \kappa = \frac{1}{4GM}$$

$$S = \frac{8\pi^3}{45h}\frac{(2GM)^4}{\beta^3} \quad \longrightarrow \quad h = \frac{\hbar}{720\pi M}$$

$$h_c \equiv \int_{r=r_H}^{r=r_H+h} ds = \int_{2GM}^{2GM+h} \frac{dr}{\sqrt{1-\frac{2GM}{r}}} = \sqrt{\frac{G\hbar}{90\pi}}$$

- Invarijantna udaljenost ne ovisi o veličini crne rupe, time interpretiramo da je zid svojstvo horizonta crne rupe
- Entropija je opisana doprinosom blizu horizonta → horizont daje kvantna svojstva crnoj rupi

Brick wall metoda za više redove WKB aproksimacije

- $(D+2)$-dimenzionalno statičko sfernosimetrično prostorvrijeme

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{g(r)} + r^2 d\Omega_D \qquad \kappa = \sqrt{\frac{g(r_H)}{f(r_H)}} f'(r_H) \qquad h_c = \int_{r_H}^{r_H+h} \frac{dr}{\sqrt{g(r)}}$$

$$g(r) = g'(r_H)(r - r_H) + \frac{1}{2} g''(r_H)(r - r_H)^2 + \cdots$$

$$f(r) = f'(r_H)(r - r_H) + \frac{1}{2} f''(r_H)(r - r_H)^2 + \cdots$$

$$\Longrightarrow \quad h_c = \sqrt{\frac{4h}{g'(r_H)}}$$

$$\left(\Box - \frac{m^2}{\hbar^2}\right)\Phi = 0 \qquad \Phi = e^{-\frac{iEt}{\hbar}} \left(\frac{R(r)}{r^{\frac{D}{2}}\sqrt{G(r)}}\right) Y_{\ell m_i}(\theta, \phi_i) \qquad G(r) = \sqrt{f(r)g(r)}$$

$$R''(r) + \left[\frac{V^2(r)}{\hbar^2} - \Delta(r)\right] R(r) = 0$$

$$V^2(r) = \frac{1}{G^2(r)}\left(E^2 - f(r)\left[m^2 + \left(\frac{\ell(\ell + D - 1)\hbar^2}{r^2}\right)\right]\right)$$

$$\Delta(r) = \left(\frac{G''(r)}{2G(r)}\right) - \left(\frac{G'^2(r)}{4G^2(r)}\right) + \left(\frac{D}{2r}\right)\left(\frac{G'(r)}{G(r)}\right) + \left(\frac{D(D-2)}{4r^2}\right)$$

$$R_{\ell m}(r) = \frac{c_0}{\sqrt{P(r)}} \exp\left[\frac{i}{\hbar}\int^r dr' P(r')\right]$$

$$\left(\frac{1}{\hbar^2}\right) P^2(r)[P^2(r) - V^2(r)] = \left(\frac{3}{4}\right) P'(r)^2 - \frac{1}{2} P''(r)P(r) - \Delta(r)P(r)^2$$

Viši redovi WKB aproksimacije

$$P(r) = \sum_{n=0}^{\infty} \hbar^{2n} P_{2n}(r)$$

$$P_0(r) = \pm V(r)$$

$$P_2(r) = \left(\frac{3}{8P_0(r)}\right)\left(\frac{P_0'(r)}{P_0(r)}\right)^2 - \left(\frac{P_0''(r)}{4\,P_0(r)}\right) - \left(\frac{\Delta(r)}{2P_o(r)}\right)$$

$$P_4(r) = -\left(\frac{5P_2^2(r)}{2V(r)}\right) - \left(\frac{4P_{2(r)\Delta(r)} + P_2''(r)}{4V^2(r)}\right) + \left(\frac{3P_2'(r)V'(r) - P_2(r)V''(r}{4V^3(r)}\right)$$

Sve funkcije vezane uz više redove WKB aproksimacije se mogu napisati preko početne, $P_0(r)$

$$N(E) = \sum_{n=0}^{\infty} N_{2n}(E) \qquad N_{2n}(E) = \frac{\hbar^{2n-1}}{\pi}\int_{r_H+h}^{L} dr \int_0^{\ell_{max}} d\ell(2\ell + D - 1)W(\ell)P_{2n}(r) \qquad W(\ell) = \frac{(\ell + D - 2)!}{(D-1)!\,\ell!}$$

Zahtjevom $P_{2n}^2 \geq 0 \;\rightarrow P_0^2(r) \geq 0$

$$\ell_{max}(\ell_{max} + 1) = \frac{r^2E^2}{\hbar^2 f(r)^2}$$

$$F = \sum_{n=0}^{\infty} F_{2n} \qquad S = \sum_{n=0}^{\infty} S_{2n} \qquad F_{2n} = -\int dE\,\frac{N_{2n}(E)}{e^{\beta \mathrm{E}} + 1} \qquad S_{2n} = \beta^2 \frac{\partial F_{2n}}{\partial \beta}$$

Entropija do 4. reda za 4D crnu rupu

- Računamo entropiju do 4. reda za 4-dimenzionalnu crnu rupu ($D = 2$), uz $f(r) = g(r)$, te $m = 0$
- Nulti red

$$P_0(r) = \frac{1}{g(r)}\left[E^2 - \frac{g(r)\hbar^2\ell(\ell+1)}{r^2}\right]^{\frac{1}{2}} \qquad N_0(E) = \frac{1}{\hbar\pi}\int_{r_H+h}^{L} dr \int_0^{\ell_{max}} d\ell(2\ell+1)P_0(r)$$

$$N_0(E) = \frac{2E^3}{3\hbar^3}\int_{r_H+h}^{L}\frac{r^2}{g^2(r)}dr \qquad S_0 = \frac{8\pi^3}{45\hbar^3}\frac{1}{\beta}\int_{r_H+h}^{L}\frac{r^2}{g^2(r)}dr$$

$$S_0 = \frac{r_H^2}{90h_c^2} + \left[\frac{\kappa r_H}{90} - \frac{g''(r_H)r_H^2}{360}\right]\ln\left(\frac{44r_H^2}{90h_c^2}\right)$$

- Drugi red

$$P_2(r) = \left(\frac{P_2^{(0)}(r)}{G(\varepsilon,r)}\right) + \lambda(r)\left(\frac{P_2^{(1)}(r)}{G^3(\varepsilon,r)}\right) + \lambda^2(r)\left(\frac{P_2^{(1)}(r)}{G^5(\varepsilon,r)}\right)$$

$$G(\varepsilon,r) = [\varepsilon - \lambda(r)]^{\frac{1}{2}} \qquad \varepsilon = E^2 \qquad \lambda(r) = \ell(\ell+1)\hbar^2 \frac{g(r)}{r^2}$$

$$P_2^{(0)}(r) = -\frac{g'(r)}{2r} \qquad P_2^{(1)}(r) = \frac{3g(r)}{4r^2} - \frac{3g'(r)}{4r} + \frac{g''(r)}{8} + \frac{g'(r)^2}{8g} \qquad P_2^{(2)}(r) = \frac{5g}{8r^2} - \frac{5g'(r)}{8r} + \frac{5g'(r)^2}{32g}$$

$$\frac{1}{G(\varepsilon,r)} = \frac{2\partial G(\varepsilon,r)}{\partial\varepsilon} \qquad \frac{1}{G^3(\varepsilon,r)} = -\frac{4\partial^2 G(\varepsilon,r)}{\partial^2\varepsilon} \qquad \frac{1}{G^5(\varepsilon,r)} = \frac{8}{3}\frac{\partial^2 G(\varepsilon,r)}{\partial^2\varepsilon}$$

$$N_2(E) = \frac{\hbar}{\pi}\int_{r_H+h}^{L} dr \int_0^{\ell_{max}} d\ell(2\ell+1)P_2(r)$$

$$\frac{\partial}{\partial x}\int_{a(x)}^{b(x)} dt f[x,t] = f[x,b(x)]\left(\frac{db(x)}{dx}\right) - f[x,a(x)]\left(\frac{da(x)}{dx}\right) + \int_{a(x)}^{b(x)} dt\left[\frac{\partial f(x,t)}{\partial x}\right]$$

$$\lambda = \ell(\ell+1)\hbar^2\frac{g(r)}{r^2}$$

$$N_2(E) = \frac{\hbar}{\pi}\int_{r_H+h}^{L} dr\int_0^{\ell_{max}} d\ell(2\ell+1)P_2(r)$$

$$\hbar N_2(E) = \frac{1}{\pi}\int_{r_H+h}^{L} dr\,\frac{r^2}{g(r)}\int_0^{\varepsilon}\left[2P_2^{(0)}(r)\frac{\partial}{\partial\varepsilon}G(\varepsilon,r) - 4P_2^{(2)}(r)\lambda\frac{\partial^2}{\partial\varepsilon^2}G(\varepsilon,r) + \frac{8}{3}P_2^{(2)}(r)\lambda^2\frac{\partial^3}{\partial\varepsilon^3}G(\varepsilon,r)\right]d\lambda$$

$$\int_0^{\varepsilon}\lambda\frac{\partial^2 G(\varepsilon,r)}{\partial\varepsilon^2}d\lambda = \frac{\partial^2}{\partial\varepsilon^2}\int_0^{\varepsilon}\lambda G(\varepsilon,r)d\lambda - \frac{\varepsilon}{2(\varepsilon-\lambda)}\Big|_{\varepsilon=\lambda}$$

WKB aproksimacija ne radi na točkama obrata!

$$N_2(E) = \frac{E}{\hbar\pi}\int_{r_H+h}^{L} dr\left[\frac{1}{3} - \frac{4rg'(r)}{3g(r)} + r^2\left(\frac{g'(r)^2}{3g(r)^2} - \frac{g''(r)}{2g(r)}\right)\right]$$

$$S_2 = \frac{\pi}{3\hbar\beta}\int_{r_H+h}^{L} dr\left[\frac{1}{3} - \frac{4rg'(r)}{3g(r)} + r^2\left(\frac{g'(r)^2}{3g(r)^2} - \frac{g''(r)}{2g(r)}\right)\right]$$

$$S_2 = \frac{11r_H^2}{90h_c^2} - \left(\frac{\kappa r_H}{10} + \frac{g''(r_H)r_H^2}{60}\right)\ln\left(\frac{44r_H^2}{90h_c^2}\right)$$

Doprinos entropiji od viših WKB modova ima isti oblik kao i doprinos vodećeg

- Četvrti red

$$P_4(r)=\left(\frac{P_4^{(0)}(r)}{G^3(\varepsilon,r}\right)+\lambda(r)\left(\frac{P_4^{(1)}(r)}{G^5(\varepsilon,r)}\right)+\lambda^2(r)\left(\frac{P_4^{(2)}(r)}{G^7(\varepsilon,r)}\right)+\lambda^3(r)\left(\frac{P_4^{(3)}(r)}{G^9(\varepsilon,r)}\right)+\lambda^4\left(\frac{P_4^{(4)}(r)}{G^{11}(\varepsilon,r)}\right)$$

$$N_4(E)=\frac{\hbar^3}{\pi}\int_{r_H+h}^{L}dr\int_0^{\ell_{max}}d\ell(2\ell+1)P_4(r)$$

$$\begin{aligned}N_4(E)\\&=-\frac{4\hbar}{\pi}\int_{r_H+h}^{L}dr\,\frac{r^2}{g(r)}P_4^{(0)}(r)\frac{\partial^2}{\partial\varepsilon^2}\int_0^{\varepsilon}G(\varepsilon,r)d\lambda+\frac{8\hbar}{3\pi}\int_{r_H+h}^{L}dr\,\frac{r^2}{g(r)}P_4^{(1)}(r)\frac{\partial^3}{\partial\varepsilon^3}\int_0^{\varepsilon}\lambda G(\varepsilon,r)d\lambda\\&-\frac{16\hbar}{15\pi}\int_{r_H+h}^{L}dr\,\frac{r^2}{g(r)}P_4^{(2)}(r)\frac{\partial^4}{\partial\varepsilon^4}\int_0^{\varepsilon}\lambda^2G(\varepsilon,r)d\lambda+\frac{32\hbar}{105\pi}\int_{r_H+h}^{L}dr\,\frac{r^2}{g(r)}P_4^{(3)}(r)\frac{\partial^5}{\partial\varepsilon^5}\int_0^{\varepsilon}\lambda^3G(\varepsilon,r)d\lambda\\&-\frac{64\hbar}{945\pi}\int_{r_H+h}^{L}dr\,\frac{r^2}{g(r)P_4^{(3)}(r)}P_4^{(3)}(r)\frac{\partial^6}{\partial\varepsilon^6}\int_0^{\varepsilon}\lambda^4G(\varepsilon,r)d\lambda\end{aligned}$$

$$N_4(E)=\frac{1}{E}\int_{r_H+h}^{L}dr\Sigma^{(4)}(r)\qquad F_4=\int_0^{\infty}dx\,\frac{1}{x(e^x+1)}\int_{r_H+h}^{L}dr\,\Sigma^{(4)}(r)\qquad S_4=0$$

$$S = S_0 + S_2 + S_4$$

$$S = \boxed{\frac{11r_H^2}{90h_c^2}} - \left(\frac{\kappa r_H}{10} + \frac{g''(r_H)r_H^2}{60}\right)\ln\left(\frac{44r_H^2}{90h_c^2}\right)$$

$$\frac{11r_H^2}{90h_c^2} = S_{BH} = \frac{A}{4\ell_{Pl}^2} \quad \Longrightarrow \quad h_c^2 = \frac{11\ell_{Pl}^2}{90\pi}$$

$$S = S_{BH} + F(A)\ln\left(\frac{A}{\ell_{Pl}^2}\right)$$

$$F(A) = -\frac{\kappa r_H}{10} - \frac{g''(r_H)r_H^2}{60}$$

Entropija za Scwharzschildovu crnu rupu

$$f(r) = g(r) = 1 - \frac{2GM}{r}$$

$$\kappa = \frac{g'(r_H)}{2} = \frac{1}{2r_H}$$

$$g''(r_H) = -\frac{2}{r_H^2}$$

$$S = S_{BH} - \frac{1}{60}\ln\left(\frac{A}{\ell_{Pl}^2}\right)$$

Entropija za BTZ crnu rupu

- BTZ crna rupa je (1 + 2)-dimenzionalna osnosimetrična crna rupa

$$ds^2 = -(N^\perp)^2 dt^2 + (N^\perp)^{-2} dr^2 + r^2\left(d\phi^2 + N^\phi dt\right)^2$$

$$N^\perp = \left(-M + \left(\frac{r}{l}\right)^2 + \left(\frac{J}{2r}\right)^2\right)^{\frac{1}{2}} \qquad N^\phi = -J/2r^2 \qquad l^{-2} = -\Lambda$$

$$r_\pm = l\left(\frac{M}{2}\left(1 \pm \sqrt{1 - \left(\frac{J}{Ml}\right)^2}\right)\right)^{\frac{1}{2}}$$

- Nerotirajuća: $J = 0$ $\qquad r_- = 0$ $\qquad r_+ = r_H$

$$S_{BH} = \frac{A}{4G\hbar} = \frac{r_H \pi}{2G\hbar} \qquad g(r) = N^{\perp} \qquad D = 1 \qquad W(\ell) = \frac{1}{\ell}$$

$$S_0 = \frac{3\zeta(3)}{4\pi^2}\frac{r_H^2}{l}\frac{1}{\sqrt{2r_H h}}$$

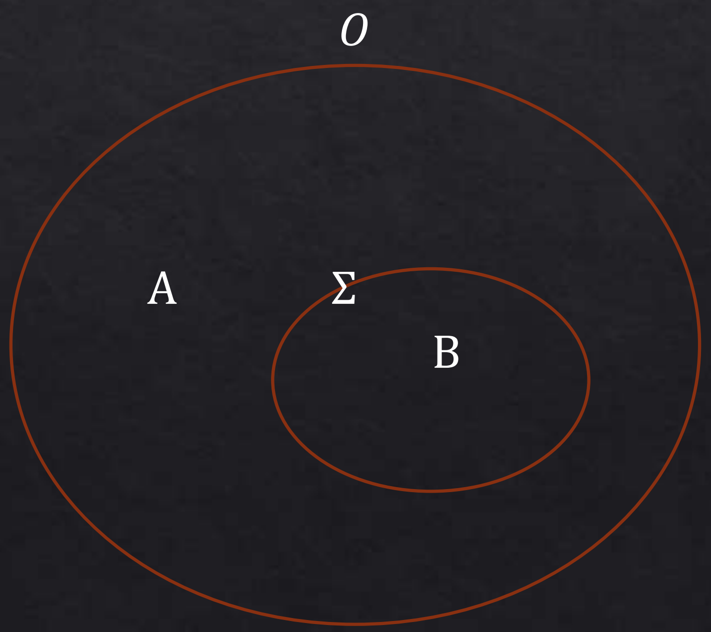
$$S_2 = \hbar\ln(2)\left(\frac{1}{\sqrt{2r_H h}}\left(-\frac{l}{2} - \frac{19}{24}\frac{l^3}{r_H^2} - \frac{5}{8}\frac{l^5}{r_H^4}\right) + \sqrt{2r_H h}\left(-\frac{3}{4}\frac{l^3}{r_H^4} - \frac{25}{32}\frac{l^5}{r_H^6}\right)\right)$$

$$S = S_{BH} + G(A)$$

$$G(A) = \frac{2\ln(2)\, G\hbar^2\left(-\frac{3}{4}\frac{l^3}{r_H^5\pi} - \frac{25}{32}\frac{l^5}{r_H^7\pi}\right)}{\left(\frac{3\zeta(3)}{4\pi^2}\frac{r_H^2}{l} + \hbar\ln(2)\left(-\frac{l}{2} - \frac{19}{24}\frac{l^3}{r_H^2} - \frac{5}{8}\frac{l^5}{r_H^4}\right)\right)}$$

Entropija sprege

- Čisto vakumsko stanje $|\psi>$ kvantnog sistema definiranog unutar područja O
- Uvodimo plohu Σ koja dijeli O na dva disjunktna podprostora A i B
- Kvantni sistem = unija dva podsistema
- $|\psi> = \sum_{i,a} \psi_{i,a} |A>_i |B>_a$
- $S = -Tr\,(\rho \ln(\rho))$



- $\rho_0(A,B) = |\psi >< \psi|$ → za čisto stanje $S = 0$
- $\rho_B = Tr_A \rho_0(A,B)$
- $S_B = -Tr(\rho_B \ln \rho_B)$ entropija sprege!
- $S_A = S_B$
- Entropija sprege za sistem u čistom stanju nije ekstenzivna veličina

- Primjena na crnim rupama: entropiju crne rupe možemo izračunati tako da izračunamo entropiju izvan nje

- Glavni doprinos entropije dolazi od samog horizonta crne rupe

- Schwarzschildova crna rupa: $S = S_{BH} + \frac{1}{90} \ln\left(\frac{A}{\ell_{Pl}^2}\right)$

- Ista struktura UV divergencije kao i entropija dobivena *brick wall* metodom

Zaključak

- Uveli smo *brick wall* oko horizonta Schwarzschildove crne rupe kako bismo odrezali divergenciju gustoće stanja na horizontu
- *Brick wall* je svojstvo horizonta same crne rupe
- Horizont daje glavni doprinos kvantnim svojstvima crne rupe
- Odredili smo korekciju entropije do 4. reda u WKB aproksimaciji
- Doprisnos entropiji od viših WKB modova ima istu strukturu kao doprinos vodećeg moda.
- Entropija Swcharzschidlove crne rupe iznosi $S = S_{BH} - \frac{1}{60} \ln\left(\frac{A}{\ell_{Pl}^2}\right)$
- Entropija nerotirajuće BTZ crne rupe ima oblik $S = S_{BH} + G(A)$
- Korekcije entropije od *brick wall* metode imaju istu UV strukturu kao i one dobivene putem sprezanja stupnjeva slobode