TEORIJA KLASIČNE ELEKTRODINAMIKE KOJA DOPUŠTA ISKLJUČIVO KVANTIZIRANE TOČKASTE NABOJE

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MOTIVACIJA

- Maxwellove jednadžbe dopuštaju kontinuirane raspodjele naboja
- Želimo jednadžbe kojima su singularnost i kvantizacija naboja *intrinzični*

  Analogija: spin u Schrödingerovoj i Diracovoj jednadžbi
- Želimo da obje teorije daju ista elektromagnetska polja

$$\nabla \cdot \boldsymbol{E}_M = \frac{\rho}{\epsilon_0}$$

$$\nabla \cdot \boldsymbol{B}_M = 0$$

$$\nabla \times \boldsymbol{E}_M = -\frac{\partial \boldsymbol{B}_M}{\partial t}$$

$$\nabla \times \boldsymbol{B}_M = \mu_0 \boldsymbol{j} + \frac{1}{c^2}\frac{\partial \boldsymbol{E}_M}{\partial t}$$

UVOD: POLJA TOČKASTOG NABOJA

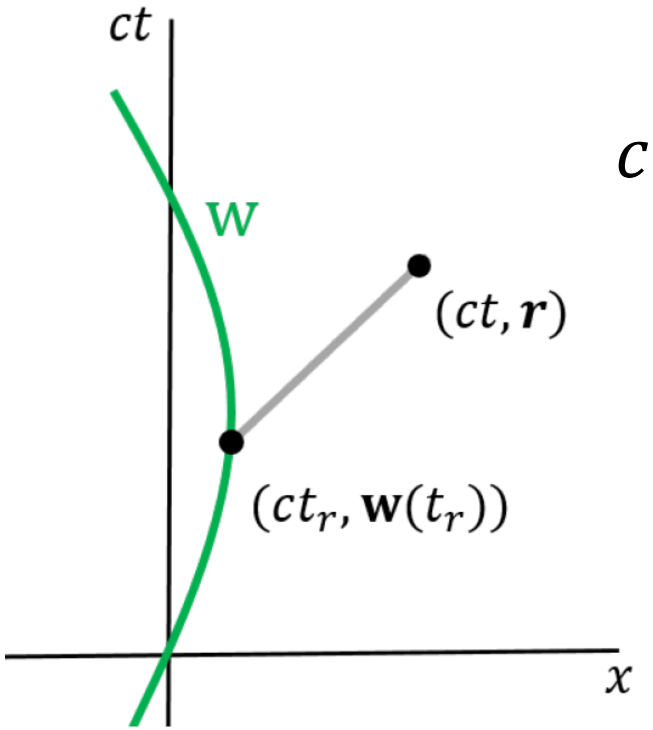
Izvori polja:

$$\rho = q\delta(\boldsymbol{r} - \boldsymbol{w}(t))$$
$$\boldsymbol{j} = q\boldsymbol{v}(t)\delta(\boldsymbol{r} - \boldsymbol{w}(t))$$

Lorentzovo baždarenje

Liénard-Wiechertovi potencijali

$$V_M = \frac{q}{4\pi\epsilon_0}\frac{1}{|\boldsymbol{s}| - \boldsymbol{s}\cdot\boldsymbol{\beta}}, \quad \boldsymbol{A}_M = \frac{1}{c}\boldsymbol{\beta}V_M$$



Retardirano vrijeme

$$c(t - t_r) = |\mathbf{r} - \mathbf{w}(\mathrm{t_r})|$$

Pokrate

$$\mathbf{s} \equiv \mathbf{r} - \mathbf{w}(\mathrm{t_r})$$
$$\boldsymbol{\beta} \equiv \frac{1}{c}\frac{d\boldsymbol{w}}{dt}\bigg|_{t_r}$$

Elektromagnetska polja

$$\boldsymbol{E}_M = -\nabla V_M - \frac{\partial \boldsymbol{A}_M}{\partial t}$$
$$\boldsymbol{B}_M = \nabla \times \boldsymbol{A}_M$$

UVOD: BERRYJEVA KONEKSIJA I ZAKRIVLJENOST

Općeniti Hamiltonijan

$$i\hbar \frac{\partial |\psi\rangle}{\partial t} = H\big(\boldsymbol{R}(t)\big)|\psi\rangle$$

Zamrznuti Hamiltonijan

$$E_n|\phi_n\rangle = H_0(\boldsymbol{R}(t))|\phi_n\rangle$$

Adijabatska aproksimacija

Evolucija sustava

$$|\psi\rangle = e^{\theta_n^D + \theta_n^G}|\phi_n(t)\rangle$$

Dinamička faza: $\theta_n^D = -\frac{i}{\hbar}\int_0^t E_n(t')dt'$

Geometrijska faza: $\theta_n^G = \int_{\boldsymbol{R}(0)}^{\boldsymbol{R}(t)} \underbrace{i\langle\psi|\boldsymbol{\nabla}_R|\psi\rangle}_{\text{Berryjeva koneksija } \boldsymbol{A}} \cdot d\boldsymbol{R}$

Berryjeva koneksija $\boldsymbol{A}$

Zatvorena krivulja u 3D parametarskom prostoru:

$$\theta_n^G = \oint_{\partial S} \boldsymbol{A} \cdot d\boldsymbol{R} = \oiint_S \underbrace{\nabla \times \boldsymbol{A}}_{\text{Berryjeva zakrivljenost } \boldsymbol{\Omega}} \cdot d\boldsymbol{a}$$

Berryjeva zakrivljenost $\boldsymbol{\Omega}$

UVOD: BERRYJEVA KONEKSIJA I ZAKRIVLJENOST

PRIMJER:

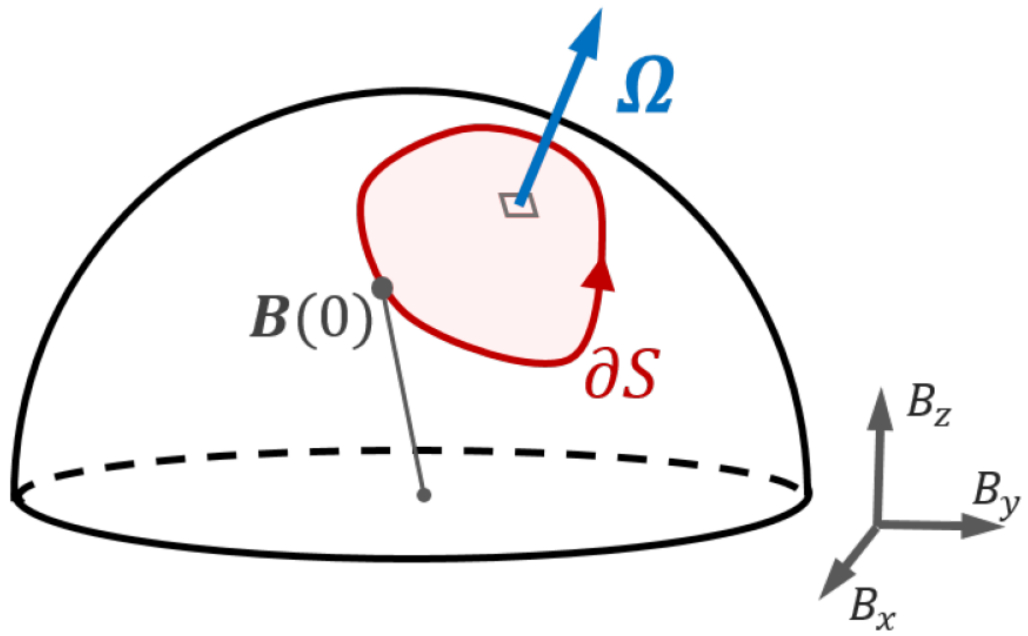
- interakcija magnetskog dipola s homogenim $B(t)$ promjenjivog smjera

$$H = \kappa \boldsymbol{\sigma} \cdot \boldsymbol{B}$$

- Stacionarni Hamiltonijan ima dva svojstvena stanja: $|\phi_+(\boldsymbol{B})\rangle, |\phi_-(\boldsymbol{B})\rangle$

- Berryjeva zakrivljenost:

$$\boldsymbol{\Omega} = \pm \frac{1}{2} \frac{\boldsymbol{B}}{|\boldsymbol{B}|^3}$$



Općenite definicije

$$A_\mu = i\langle\psi|\partial_\mu|\psi\rangle$$

$$\Omega_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

UVOD: WEYLOVA JEDNADŽBA

Bezmasena Diracova jednadžba

$$\gamma^\mu \frac{\partial}{\partial x^\mu} \tilde{\psi}(x^\nu; k^\mu) = 0$$

Ansatz (ravni val) →

Algebarski oblik jednadžbe

$$\gamma^\nu k_\nu \psi(k^\mu) = 0$$

Weylova (kiralna) reprezentacija: $\gamma^0 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}$ $\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$

Diracova jednadžba cijepa se na lijevu i desnu Weylovu jednadžbu:

$$\begin{pmatrix} 0 & \mathbf{1}k^0 - \boldsymbol{\sigma} \cdot \boldsymbol{k} \\ \mathbf{1}k^0 + \boldsymbol{\sigma} \cdot \boldsymbol{k} & 0 \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = 0$$

FORMULACIJA TEORIJE: WEYLOVA JEDNADŽBA

- Teorija je formulirana pomoću para Weylovih jednadžbi:

$$\left[\mathbf{1}\frac{\partial}{\partial k^0} \pm \sigma^i \frac{\partial}{\partial k^i}\right]\tilde{\psi}_{R,L}(k^\nu;\rho^\mu) = 0$$

Ansatz

$$\tilde{\psi}(k^\nu;\rho^\mu) = \psi(\rho^\mu)e^{-ik^\nu\rho_\nu}$$

$$\boldsymbol{\sigma}\cdot\boldsymbol{\rho}\,\psi_{R,L}(\rho^\mu) = \pm\rho^0\psi_{R,L}(\rho^\mu)$$

- $\tilde{\psi}(k^\nu;\rho^\mu)$ parametarski ovisi o ρ^μ, koji sadrži informaciju o trajektoriji naboja

Analogija

- "obična" bezmasena Diracova jednadžba:

$$\left[\gamma^0\frac{\partial}{\partial x^0} + \gamma^i \frac{\partial}{\partial x^i}\right]\tilde{\psi}(x^\nu;k^\mu) = 0$$

Ansatz

$$\tilde{\psi}(x^\nu;k^\mu) = \psi(k^\mu)e^{-ix^\nu k_\nu}$$

$$\gamma^i k_i\psi(k^\mu) = -\gamma^0 k_0\psi(k^\mu)$$

- $\tilde{\psi}(x^\nu;k^\mu)$ parametarski ovisi o k^μ

FORMULACIJA TEORIJE: DEFINICIJA ρ^μ

- Desna Weylova jednadžba: $\boldsymbol{\sigma} \cdot \boldsymbol{\rho}\, \psi_R(\rho^\mu) = \rho^0 \psi_R(\rho^\mu)$

- ρ^μ sadrži ovisnost o prostoru, vremenu i trajektoriji naboja

$$\rho^\mu \equiv \Lambda^\mu{}_\nu (x^\nu - w^\nu) = \Lambda^\mu{}_\nu\, s^\nu$$

$$x^\nu \equiv (ct, \boldsymbol{r})$$
$$w^\nu \equiv (ct_c, \boldsymbol{w}(t_c))$$

LT oblika $\boldsymbol{\Lambda} = \boldsymbol{R} \cdot \boldsymbol{B}$ ovisi o brzinama i akceleracijama izvrijednjenim u t_c

- Koje su dozvoljene vrijednosti t_c?

$\rho^0 = \pm|\boldsymbol{\rho}|$ — Veza preko LT →

$s^0 = \pm|\boldsymbol{s}|$

$sgn(\rho^0) = sgn(s^0)$

$$t_c = t \pm \frac{1}{c}|\boldsymbol{r} - \boldsymbol{w}(t_c)|$$

FORMULACIJA TEORIJE: DEFINICIJA ρ^μ

- Desna Weylova jednadžba: $\boldsymbol{\sigma} \cdot \boldsymbol{\rho}\, \psi_R(\rho^\mu) = \rho^0 \psi_R(\rho^\mu)$

$$t_c = t \pm \frac{1}{c} |\boldsymbol{r} - \boldsymbol{w}(t_c)|$$

Za $\rho^0 > 0$ se dobiva retardirano rješenje $\psi_{R,p}$
Za $\rho^0 < 0$ se dobiva avansirano rješenje $\psi_{R,n}$

- Iz lijeve Weylove jednadžbe se analogno dobivaju $\psi_{L,p}, \psi_{L,n}$

- **Zahtjev kauzalnosti:** prihvatljiva samo retardirana rješenja

$$\psi_{R,p} \equiv \psi_- \text{ i } \psi_{L,p} \equiv \psi_+$$

FORMULACIJA TEORIJE: DEFINICIJA ρ^μ

- $\rho^\mu \equiv \Lambda^\mu{}_\nu (x^\nu - w^\nu) = \Lambda^\mu{}_\nu\, s^\nu \quad , \quad \boldsymbol{\Lambda} = \boldsymbol{R} \cdot \boldsymbol{B}$

- Lorentzov potisak $\boldsymbol{B}$ definiran je brzinom $\boldsymbol{v}(t_r)$

- Matrica rotacije definirana je $\boldsymbol{R} \equiv \exp(i\boldsymbol{\theta} \cdot \boldsymbol{J})$

Generatori rotacije

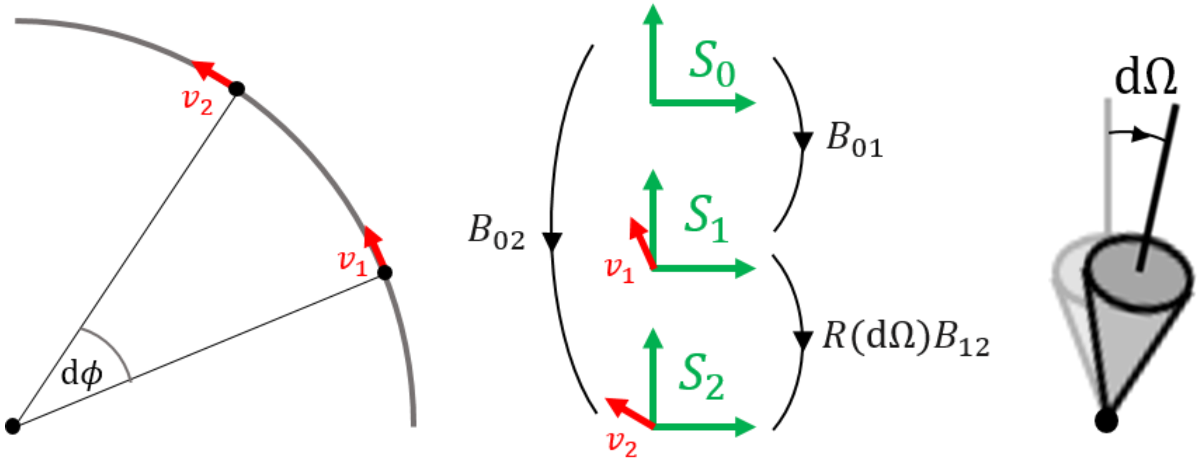
$$\boldsymbol{J} = (J_x, J_y, J_z)$$

$$\boldsymbol{\theta} \equiv \hat{n}(t_r) \cdot \int_a^{t_r} [\boldsymbol{\omega}_{Th}(t') \cdot \hat{n}(t_r)] dt'$$

Thomasova frekvencija

$$\boldsymbol{\omega}_{Th}(t') = \frac{1}{c^2} \frac{\gamma^2}{\gamma + 1} \boldsymbol{a}(t') \times \boldsymbol{v}(t')$$

Thomasova precesija



FORMULACIJA TEORIJE: EM POLJA

Weylove jednadžbe

$$\boldsymbol{\sigma} \cdot \boldsymbol{\rho}\, |\psi_+\rangle = -\, \rho^0 |\psi_+\rangle$$

$$\boldsymbol{\sigma} \cdot \boldsymbol{\rho}\, |\psi_-\rangle = +\, \rho^0 |\psi_-\rangle$$

Generatorske funkcije

$$|\psi_+\rangle = \begin{pmatrix} \sin\frac{\theta}{2} e^{-i\phi} \\ -\cos\frac{\theta}{2} \end{pmatrix}, |\psi_-\rangle = \begin{pmatrix} \cos\frac{\theta}{2} e^{-i\phi} \\ sin\frac{\theta}{2} \end{pmatrix}$$

$$\rho^1 = |\boldsymbol{\rho}| sin\theta cos\phi$$

$$\rho^2 = |\boldsymbol{\rho}| sin\theta sin\phi$$

$$\rho^3 = |\boldsymbol{\rho}| cos\theta$$

Potencijali

(Berryjeva koneksija)

$$V = \frac{1}{c^2}\left\langle\psi\middle|\frac{\partial}{\partial t}\middle|\psi\right\rangle, \quad \boldsymbol{A} = \langle\psi|\boldsymbol{\nabla}|\psi\rangle$$

EM polja točkastog naboja

(Berryjeva zakrivljenost)

$$\boldsymbol{E} = \nabla \times \boldsymbol{A}, \quad \boldsymbol{B} = \boldsymbol{\nabla} V + \frac{1}{c^2}\frac{\partial \boldsymbol{A}}{\partial t}$$

FORMULACIJA TEORIJE: EM POLJA

- Ova teorija daje ista polja kao i Maxwellove jednadžbe, do na konstantu:

$$\boldsymbol{E}_M = \frac{e}{2\pi\epsilon_0}\boldsymbol{E}, \qquad \boldsymbol{B}_M = \frac{e}{2\pi\epsilon_0}\boldsymbol{B}$$

- Od kuda dolazi **singularnost**?

$$\boldsymbol{E} = \nabla \times \boldsymbol{A} \longrightarrow \nabla \cdot \boldsymbol{E} = \nabla \cdot (\nabla \times \boldsymbol{A}) \stackrel{?}{=} 0$$

Jednakost vrijedi u svim točkama osim singulariteta

Volumna gustoća naboja nužno iščezava, jedini mogući naboji su singulariteti

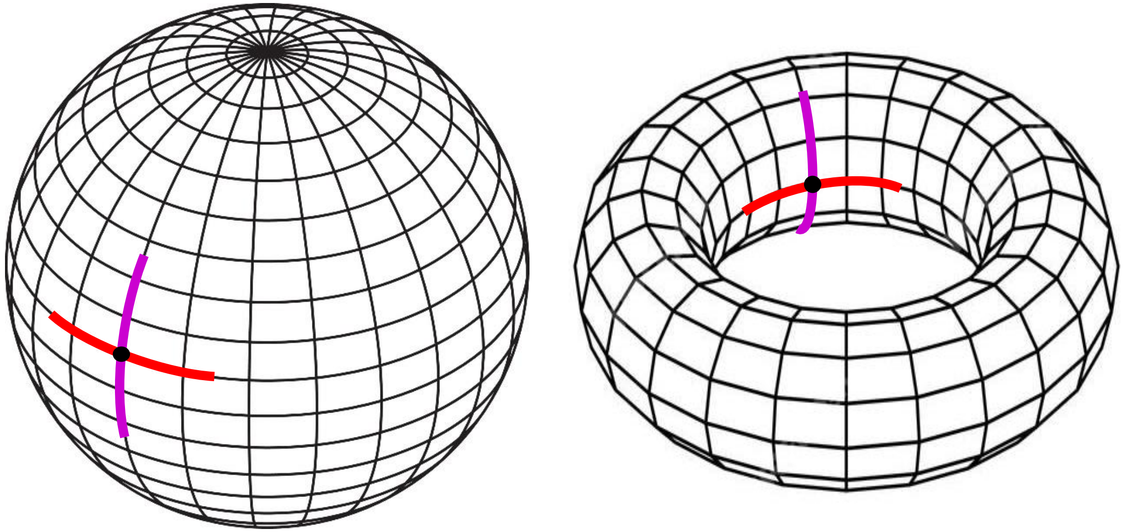
- **Princip superpozicije** - generalizacija na proizvoljni broj naboja

FORMULACIJA TEORIJE: KVANTIZACIJA

Gauss-Bonnet teorem (GB)

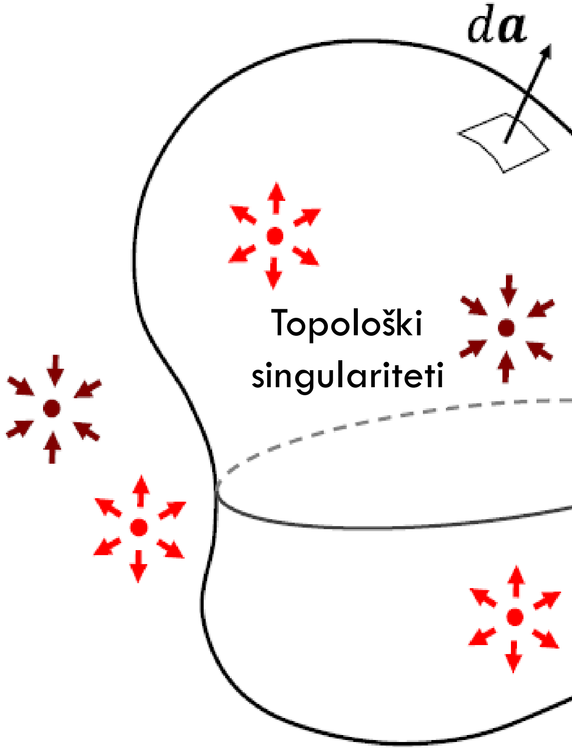
Gaussova zakrivljenost: $\Omega_G = \kappa_{min} \cdot \kappa_{max}$

$$\oiint_S \Omega_G \cdot da = 2\pi m$$



$m = 2$ $m = 0$

- **Chernov teorem** – generalizacija GB



$\boldsymbol{E}$ odgovara Berryjevoj zakrivljenosti

$$\frac{1}{2\pi} \oiint_S \boldsymbol{E} \cdot d\boldsymbol{a} = m \in \mathbb{Z}$$

→ **kvantizacija**

PRIMJERI: GIBANJE PO PRAVCU

- Weylovu jednadžbu možemo gledati kao eigenproblem:

$$H\psi_R(\rho^\mu) = \rho^0\psi_R(\rho^\mu), \qquad H = \boldsymbol{\sigma} \cdot \boldsymbol{\rho}$$

Gibanje po pravcu → $a \parallel v$

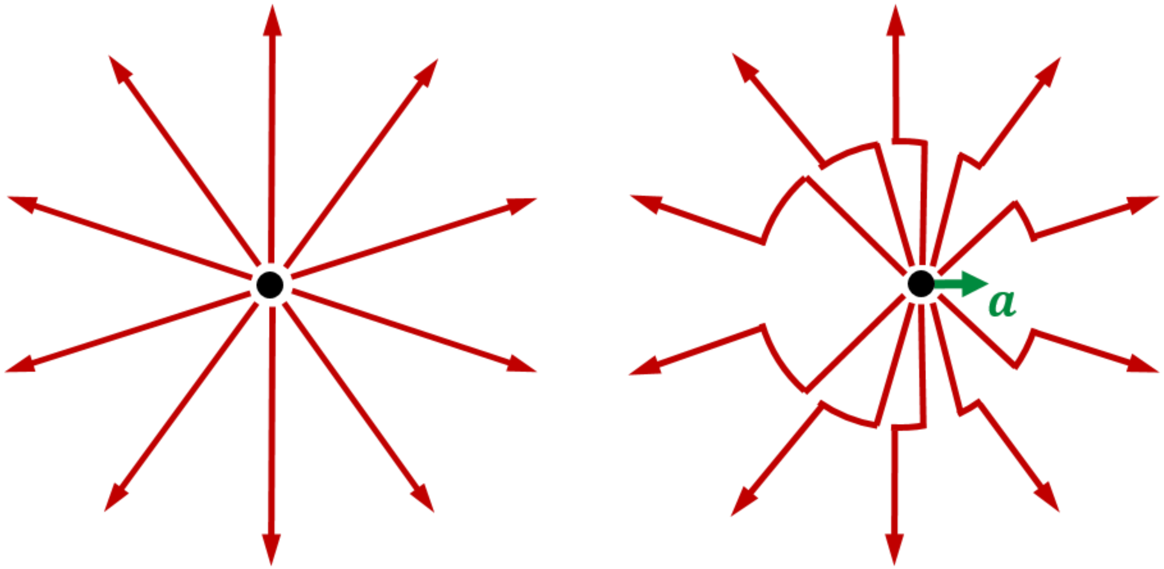
Nema Thomasove precesije

a) Mirujući naboj

$$H = x\sigma_x + y\sigma_y + z\sigma_z$$

b) Gibanje konstantnom brzinom

$$H = x\sigma_x + y\sigma_y + \gamma(z - v_z t)\sigma_z$$



c) Općenito 1D gibanje

$$H = x\sigma_x + y\sigma_y + \gamma(t_r)[z(t_r) - v_z(t_r)(t - t_r)]\sigma_z$$

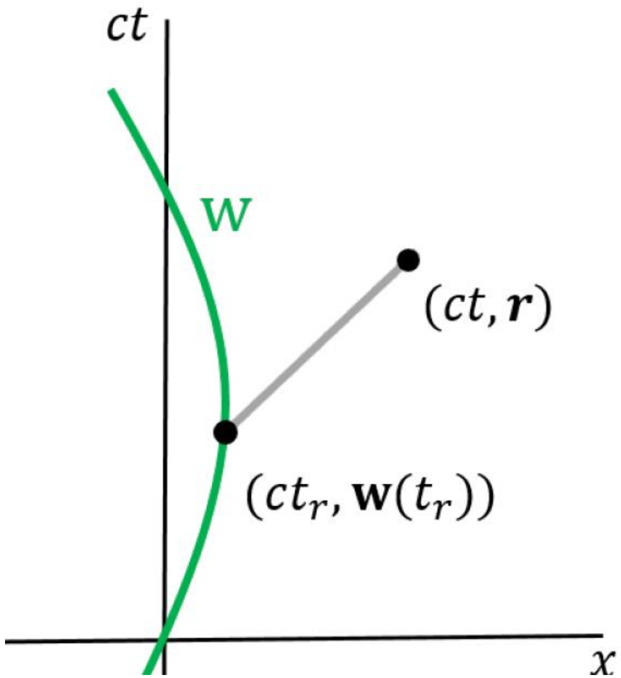
PRIMJERI: OPĆENITO GIBANJE U RAVNINI

- Općenito gibanje u $x - y$ ravnini:

$$\begin{bmatrix}\rho^0\\ \rho^1\\ \rho^2\\ \rho^3\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & \cos\theta_z & \sin\theta_z & 0\\ 0 & -\sin\theta_z & \cos\theta_z & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\gamma & -\gamma\beta_x & -\gamma\beta_y & 0\\ -\gamma\beta_x & 1+(\gamma-1)\beta_x^2/\boldsymbol{\beta}^2 & (\gamma-1)\beta_x\beta_y/\boldsymbol{\beta}^2 & 0\\ -\gamma\beta_y & (\gamma-1)\beta_x\beta_y/\boldsymbol{\beta}^2 & 1+(\gamma-1)\beta_x^2/\boldsymbol{\beta}^2 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}c(t-t_r)\\ x-w_x(t_r)\\ y-w_y(t_r)\\ z\end{bmatrix}$$

- Polje u točki x^μ ovisi samo o jednoj točki na svjetskoj liniji naboja – nije bitna povijest gibanja

$$\theta_z = \frac{1}{c^2}\int_a^{t_r}\frac{\gamma^2}{\gamma+1}\left(a_x v_y - a_y v_x\right)dt'$$



- Konačna polja ne smiju ovisiti o θ_z, već samo o njegovoj derivaciji $\omega_{Th}(t_r)$

PRIMJERI: OPĆENITO 3D GIBANJE

- Polja u x^μ ovise samo o jednoj točki svjetske linije
- Svaka 3D putanja lokalno "izgleda" kao 2D

Općeniti 3D i 2D Hamiltonijani su gotovo identični

Jedina razlika 3D i 2D slučaja je definicija $\boldsymbol{\theta}$

$$\boldsymbol{\theta}_{\mathbf{2D}}(t_r) = \int_a^{t_r} \boldsymbol{\omega}_{Th}(t')dt'$$

$$\boldsymbol{\theta}_{\mathbf{3D}}(t_r) \equiv \hat{n}(t_r) \cdot \int_a^{t_r} [\boldsymbol{\omega}_{Th}(t') \cdot \hat{n}(t_r)]dt'$$

- Kod općenite 3D putanje $\boldsymbol{\omega}_{Th}$ mijenja smjer u vremenu
- U 3D definiciji je nužna projekcija kako bi vrijedilo $\boldsymbol{\theta}_{\mathbf{3D}}(t_r) \parallel \boldsymbol{\omega}_{Th}(t_r)$

HVALA NA POZORNOSTI