Fidelity approach to frustrated quantum XY model

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What are we talking about?

- XY chain
- Quantum phase diagram
- Frustration
- Fidelity

XY chain

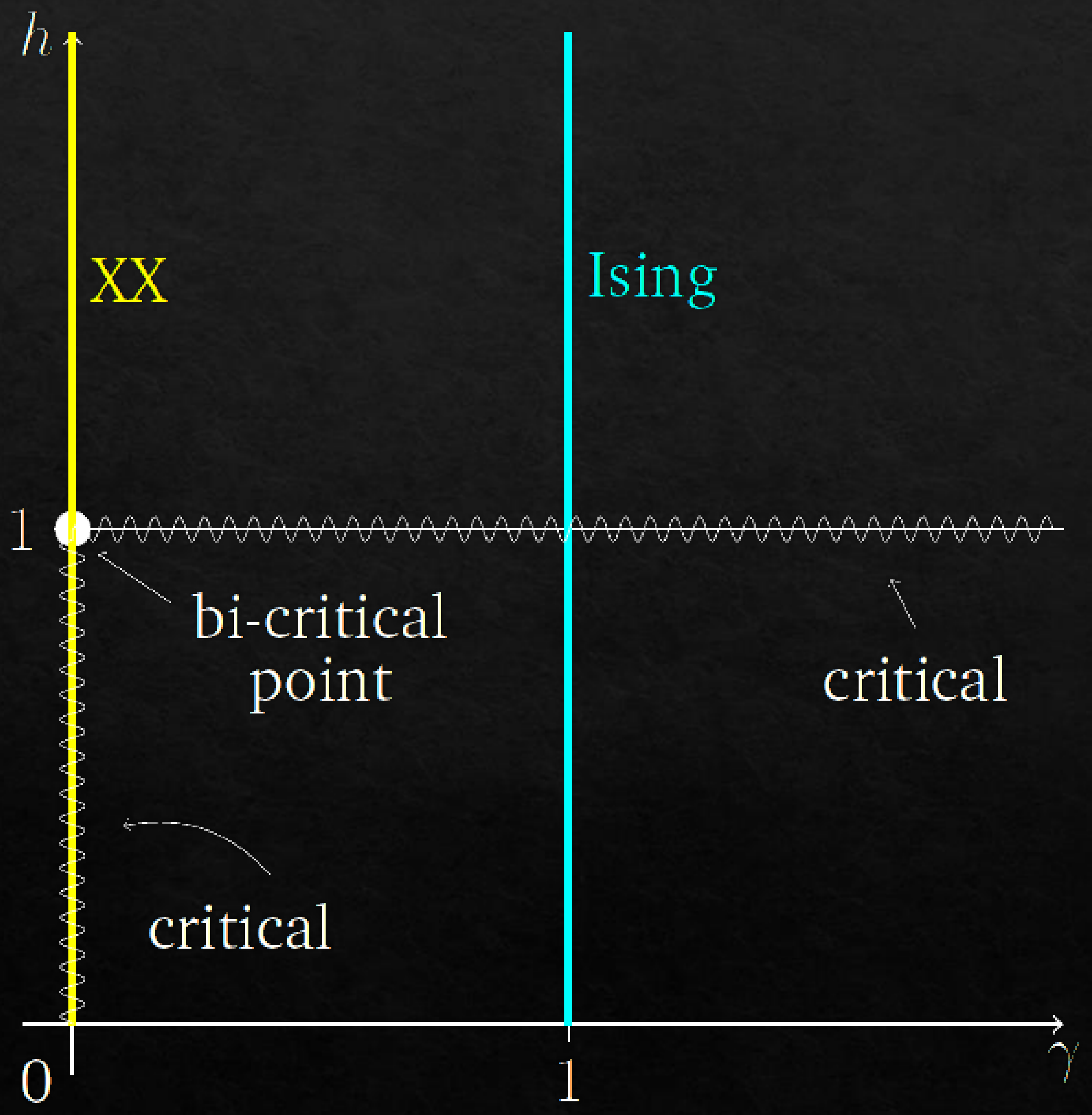
$$H = \frac{J}{2}\sum_{j=1}^{N}\left(\frac{1+\gamma}{2}\sigma_j^x\sigma_{j+1}^x + \frac{1-\gamma}{2}\sigma_j^y\sigma_{j+1}^y + h\sigma_j^z\right)$$

γ – anisotropy parameter

h – external magnetic field

J – coupling constant

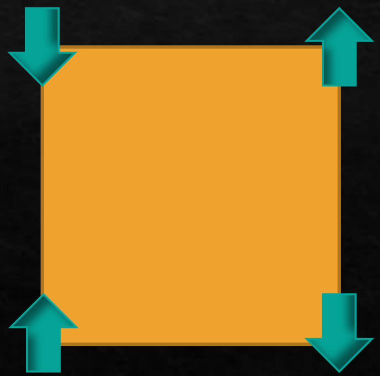
Phase diagram



$$H = \frac{J}{2}\sum_{j=1}^{N}\left(\frac{1+\gamma}{2}\sigma_j^x\sigma_{j+1}^x + \frac{1-\gamma}{2}\sigma_j^y\sigma_{j+1}^y + h\sigma_j^z\right)$$

Frustration

- Odd number of spins N
- Periodic boundary conditions (i.e., *closed* chain) $\psi_{N+1} = \psi_1$
- Antiferromagnetic chain J=1



Fidelity

- Overlap function

$$F(\rho,\sigma) = \mathrm{tr}\left(\sqrt{\rho^{1/2}\sigma\rho^{1/2}}\right) \xrightarrow[\text{states}]{\text{pure}} F(Z,\tilde{Z}) = |\langle\psi_Z|\psi_{\tilde{Z}}\rangle|$$

- Between GS of slightly different Hamiltonian **(δh, δγ)**
- Usually **F≈1**, at critical points **F** drops suddenly

Solving the XY chain

$$H = \frac{J}{2}\sum_{j=1}^{N}\left(\frac{1+\gamma}{2}\sigma_j^x\sigma_{j+1}^x + \frac{1-\gamma}{2}\sigma_j^y\sigma_{j+1}^y + h\sigma_j^z\right)$$

Jordan-Wigner → Fourier transform → Bogoliubov

Jordan-Wigner

- mapping spins to fermions

$$\psi_j = \left(\prod_{l=1}^{j-1}\sigma_l^z\right)\sigma_j^+$$

- separating into parity sectors

$$P\,|\text{even }n(\psi_j)\rangle = +\,|\text{even }n(\psi_j)\rangle$$
$$P\,|\text{odd }n(\psi_j)\rangle = -\,|\text{odd }n(\psi_j)\rangle$$

Fourier transform

- moving into q-space

$$\psi_q = \frac{1}{\sqrt{N}}\sum_{l=1}^{N}\psi_l e^{-i\frac{2\pi}{N}ql}$$

Bogoliubov

- rotation of phase space
- leads to diagonal H

→ elementary excitations

$$\begin{pmatrix}\cos\theta_q & -\sin\theta_q\\ \sin\theta_q & \cos\theta_q\end{pmatrix}\begin{pmatrix}\psi_q\\ \psi_{-q}^\dagger\end{pmatrix} = \begin{pmatrix}\chi_q\\ -\chi_{-q}^\dagger\end{pmatrix}$$

$$H = \frac{J}{2}\sum_{j=1}^{N}\left[(\sigma_j^+\sigma_{j+1}^- + \gamma\sigma_j^+\sigma_{j+1}^+ + \text{h.c.}) + h\sigma_j^z\right] = \frac{1+P}{2}H^+ + \frac{1-P}{2}H^-$$

$$H^\pm = -J\sum_q \Lambda_q\left(\chi_q^\dagger\chi_q - \frac{1}{2}\right)$$

$$\chi_q = \cos\theta_q\psi_q - \sin\theta_q\psi_{-q}^\dagger$$

$$\Lambda_q = \sqrt{\left[h - \cos\left(\frac{2\pi}{N}q\right)\right]^2 + \gamma^2\sin^2\left(\frac{2\pi}{N}q\right)}$$

$$|GS^+\rangle = \prod_{q=0}^{\lfloor\frac{N}{2}\rfloor-1}\left(\cos\theta_{q+1/2} + \sin\theta_{q+1/2}\,\psi_{q+1/2}^\dagger\psi_{q+1/2}^\dagger\right)|0\rangle$$

$$|GS^-\rangle = \psi_{q=0}^\dagger\prod_{q=0}^{\lfloor\frac{N-1}{2}\rfloor}\left(\cos\theta_q + \sin\theta_q\psi_q^\dagger\psi_{-q}^\dagger\right)|0\rangle$$

Fidelity, again

$$F(Z, \tilde{Z}) = \left| \det \frac{T + \tilde{T}}{2} \right|^{1/2}$$

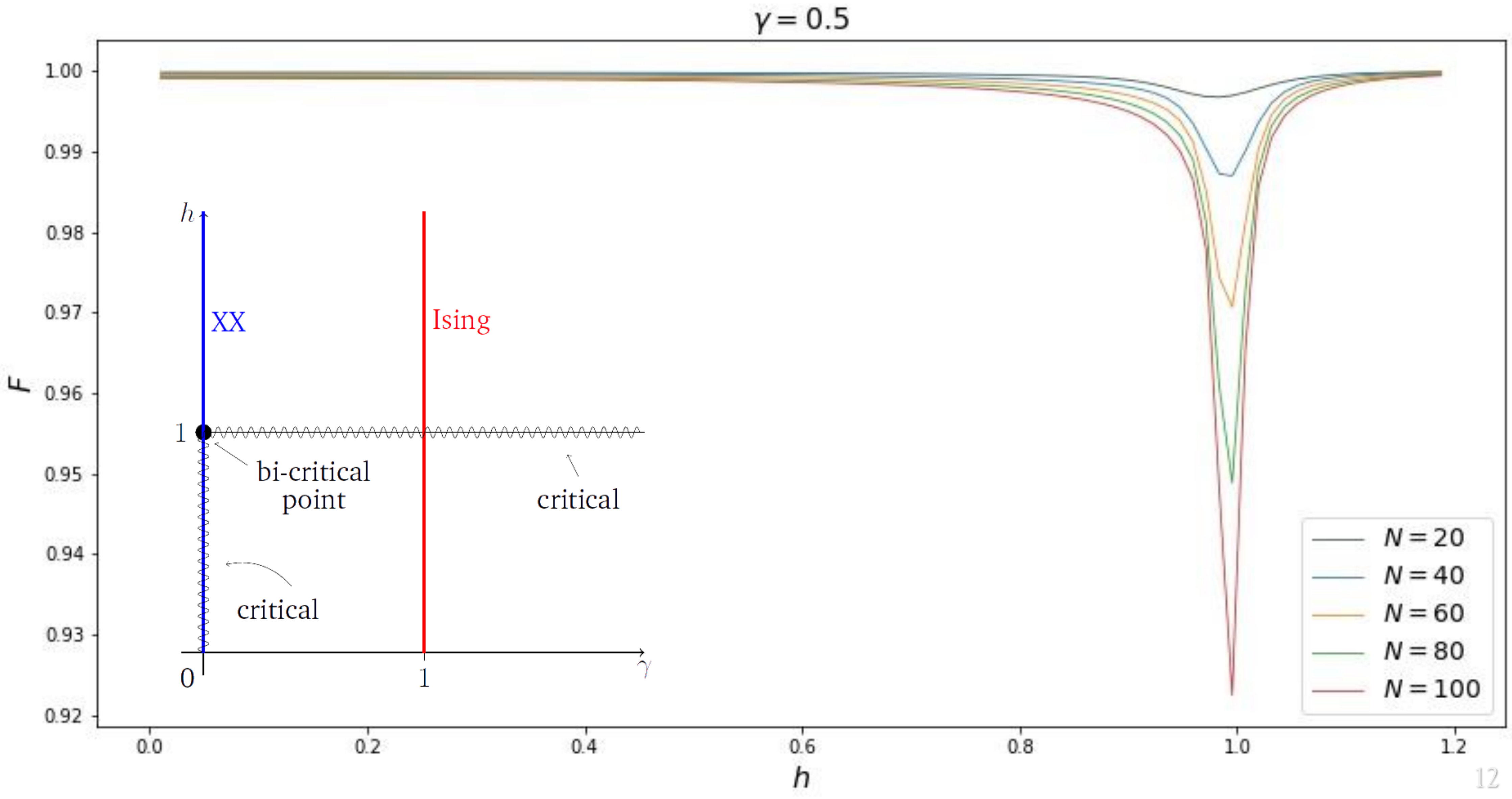
$$T = f(\Lambda, \text{ eigenvectors})$$

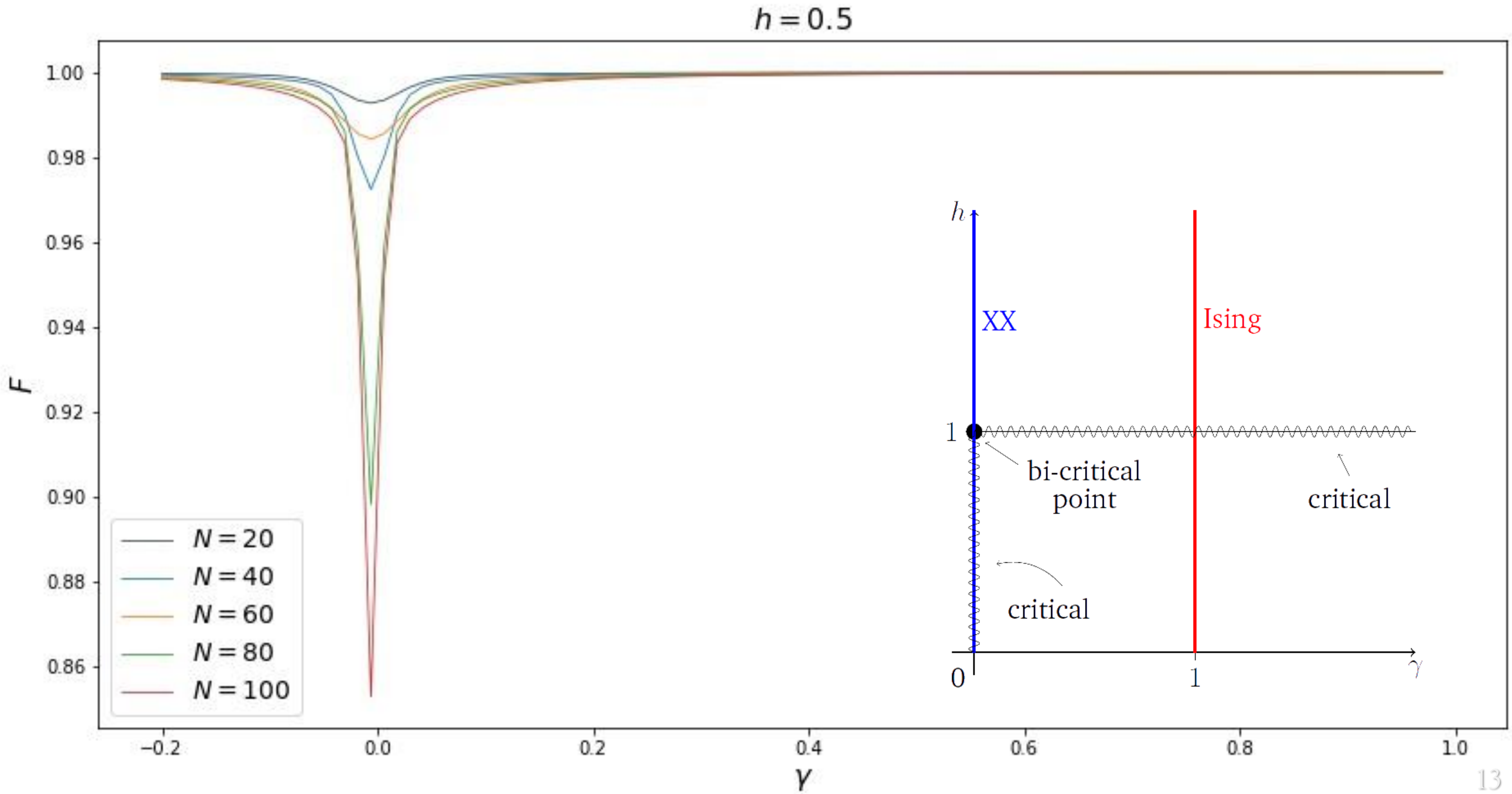
Fidelity, again

$$F(Z,\tilde{Z}) = \left| \det \frac{T+\tilde{T}}{2} \right|^{1/2}$$

$$|\psi_Z\rangle = \exp\left(\frac{1}{2}\sum_{i,j=0}^{N} c_i^\dagger G_{ij} c_j^\dagger\right)|0\rangle \rightarrow G = \frac{T-1}{T+1} \rightarrow F(Z,\tilde{Z}) = \frac{\det\left(\mathbb{I}+G^\dagger\tilde{G}\right)^{1/2}}{\det\left(\mathbb{I}+G^\dagger G\right)^{1/4}\det\left(\mathbb{I}+\tilde{G}^\dagger\tilde{G}\right)^{1/4}}$$

Results


$\gamma = 0.5$
1.00
0.99
0.98
0.97
0.96
0.95
0.94
0.93
0.92
F
0.0
0.2
0.4
0.6
0.8
1.0
1.2
h
$N = 20$
$N = 40$
$N = 60$
$N = 80$
$N = 100$
h
XX
Ising
1
bi-critical point
critical
critical
0
1
γ



h = 0.5
1.00
0.98
0.96
0.94
0.92
0.90
0.88
0.86
F
−0.2
0.0
0.2
0.4
0.6
0.8
1.0
γ
N = 20
N = 40
N = 60
N = 80
N = 100
h
XX
Ising
1
bi-critical
point
critical
critical
0
1
γ

Conclusion

- We have analyzed the ground-state fidelity for the unfrustrated 1D XY model
- Fidelity matches expectations → successful identification of critical points
- To be continued… → expand the analysis for frustrated systems

References

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