Disperzijske relacije iz nekomutativne geometrije te primjena i učinak na vrijeme dolaska fotona različitih energija

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Motivacija

Simetrijska stuktura i Hopfova algebra

Drinfel'dovo zakretanje i deformacija

κ-prostorvrijeme, algebra diferencijalnih formi i valna jednadžba

Disperzijske relacije i posljedice

Motivacija

Nekomutativnost - geometrija preko algebre glatkih funkcija

$C^\infty(M) \cong C^\infty(N) \longrightarrow M \cong N$

Kvantna mehanika (Gelfland-Naimark theorem)

$[x_i, p_j] = i\hbar\delta_{ij}$

Nekomutativna geometrija

$[x^\mu, x^\nu] = \lambda F^{\mu\nu}(x)$

Sadržaj

Simetrijska struktura

Simetrijska struktura - skup transformacija T sa preslikavanjima na tom skupu

$$\Delta(T) = \sum_i T^i_{(1)} \bigotimes T^i_{(2)} \qquad \epsilon(T) \in \mathbb{C} \qquad S(T) \in T$$

Akcija $\triangleright$ - djelovanje transformacija na (UA) reprezentaciju R_T

$$T_1 \triangleright T_2 \triangleright f = (T_1 T_2) \triangleright f$$

$$\mathbf{1} \triangleright f = f$$

$$T_1 \triangleright T_2 \triangleright T_3 \triangleright f = (T_1 T_2) \triangleright T_3 \triangleright f = T_1 \triangleright (T_2 T_3) \triangleright f$$

$$T \triangleright fg = (T_{(1)} \triangleright f)(T_{(2)} \triangleright g)$$

$$T \triangleright fgh = T \triangleright (fg)h = T \triangleright f(gh)$$

$$T \triangleright 1f = T \triangleright f1 = T \triangleright f$$

$$T \triangleright 1 = \epsilon(T)$$

$$S(T_{(1)})T_{(2)} = T_{(1)}S(T_{(2)}) = \epsilon(T)\mathbf{1}$$

Univerzalna omotačka algebra $U(\mathfrak{g})$

$$U(\mathfrak{g}) := \mathrm{span}(x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n} | x_j \in \mathfrak{g},\ i_j \in \mathbb{N}_0)$$

$$x_i x_j - x_j x_i = [x_i, x_j] = C_{ijk} x_k$$

Veliki dio informacija o Lievoj grupi sadržan
Pripadna Lieva algebra u potpunosti sadržana te proširena do UA algebre

Univerzalno svojstvo

Homomorfizam $\phi : \mathfrak{g} \to \mathcal{A}$ jedinstveno se proširuje na homomorfizam $\tilde{\phi} : U(\mathfrak{g}) \to \mathcal{A}$

$U(\mathfrak{g})$ + zahtjevi nad sim. struk. = Hopfova algebra

Hopfova algebra $(H, m, \eta, \Delta, \epsilon, S)$

$$m \circ (m \otimes \mathrm{id}) = m \circ (\mathrm{id} \otimes m)$$

$$m \circ (\eta \otimes \mathrm{id}) = \mathrm{id} = m \circ (\mathrm{id} \otimes \eta)$$

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$$

$$(\epsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \epsilon) \circ \Delta$$

$$m \circ (S \otimes \mathrm{id}) \circ \Delta = \eta \circ \epsilon = m \circ (\mathrm{id} \otimes S) \circ \Delta$$

$$\Delta(ab) = \Delta(a)\Delta(b), \quad \epsilon(ab) = \epsilon(a)\epsilon(b), \quad \Delta(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1}, \quad \epsilon(\mathbf{1}) = 1$$

Kanonska Hopfova algebra na $U(\mathfrak{g})$

$$\Delta(x) = x \otimes 1 + 1 \otimes x, \quad \epsilon(x) = 0, \quad S(x) = -x, \quad x \in \mathfrak{g}$$

Sadržaj

Drinfel'dovo zakretanje

$$(U(\mathfrak{g}), m, \eta, \Delta, \epsilon, S) \longrightarrow (U^{\mathcal{F}}(\mathfrak{g}), m^{\mathcal{F}}, \eta^{\mathcal{F}}, \Delta^{\mathcal{F}}, \epsilon^{\mathcal{F}}, S^{\mathcal{F}})$$

Drinfel'dovo zakretanje dano sa invertibilnim elementom
$\mathcal{F} = \mathcal{F}_1 \otimes \mathcal{F}_2 \in U^{\mathcal{F}}(\mathfrak{g}) \otimes U^{\mathcal{F}}(\mathfrak{g})$

$$\Delta^{\mathcal{F}}(x) = \mathcal{F}\Delta(x)\mathcal{F}^{-1} \qquad S^{\mathcal{F}}(x) = \mathcal{F}_1 S(\mathcal{F}_2) S(x) S(\mathcal{F}_1'^{-1})\mathcal{F}_2'^{-1}$$

$$(\mathbf{1} \otimes \mathcal{F})(\mathrm{id} \otimes \Delta)\mathcal{F} = (\mathcal{F} \otimes \mathbf{1})(\Delta \otimes \mathrm{id})\mathcal{F}$$
$$(\mathrm{id} \otimes \epsilon)\mathcal{F} = \mathbf{1} = (\epsilon \otimes \mathrm{id})\mathcal{F}$$

Deformacija reprezentacijskih algebri

$$(\mathcal{A}, \cdot) \longrightarrow (\mathcal{A}^{\mathcal{F}}, \star)$$

Konzistentnost dana sa: $T \triangleright f \star g = (T_{(1)} \triangleright f) \star (T_{(2)} \triangleright g)$

Deformirani produkt

$$f \star g = (\mathcal{F}_1^{-1} \triangleright f) \cdot (\mathcal{F}_2^{-1} \triangleright g)$$

Deformirano djelovanje lin. operatora na $\mathcal{A}$

$$\Phi^{\mathcal{F}}(f) = \mathcal{F}_{1(1)}^{-1} \triangleright \Phi(S^{\mathcal{F}}(\mathcal{F}_{1(2)}^{-1})\mathcal{F}_2^{-1} \triangleright f))$$

Sadržaj

Konformalna algebra

$$[M_{\mu\nu}, M_{\rho\sigma}] = i(\eta_{\mu\sigma} M_{\nu\rho} + \eta_{\nu\rho} M_{\mu\sigma} - \eta_{\mu\rho} M_{\nu\sigma} - \eta_{\nu\sigma} M_{\mu\rho})$$

$$[M_{\mu\nu}, P_\sigma] = i(\eta_{\nu\sigma} P_\mu - \eta_{\mu\sigma} P_\nu)$$

$$[M_{\mu\nu}, K_\sigma] = i(\eta_{\nu\sigma} K_\mu - \eta_{\mu\sigma} K_\nu)$$

$$[D, P_\mu] = iP_\mu$$

$$[D, K_\mu] = -iK_\mu$$

$$[K_\mu, P_\nu] = 2i(\eta_{\mu\nu} D - M_{\mu\nu})$$

$$M_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)$$

$$P_\mu = -i\partial_\mu$$

$$K_\mu = -i(2x_\mu x^\nu \partial_\nu - x_\nu x^\nu \partial_\mu)$$

$$D = -ix^\mu \partial_\mu$$

κ-prostorvrijeme

$$[x^0, x^j] = \frac{i}{\kappa} x^j \qquad [x^i, x^j] = 0$$

Algebra glatkih funkcija $C^\infty(M)$ na prostorvremenu deforimrana Drinfel'dovi zakretanjem

$$\mathcal{F} = \exp\left(-iD \otimes \ln(1 + \frac{1}{\kappa} P_0)\right)$$

$$\begin{aligned}
\mathcal{F}^{-1} &= \exp\left(iD \otimes \ln(1 + \frac{1}{\kappa} P_0)\right) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} [iD(iD-1)...(iD-n+1)] \otimes \left(\frac{1}{\kappa} P_0\right)^n \\
&= 1 \otimes 1 + x^\mu \partial_\mu \otimes \frac{-i}{\kappa} \partial_0 + ...
\end{aligned}$$

Prezentacija računa

$$x^0 \star x^i = x^0 x^i + (x^\mu \partial_\mu x^0)(\frac{-i}{\kappa}\partial_0 x^i) + ... = x^0 x^i$$

$$x^i \star x^0 = x^i x^0 + (x^\mu \partial_\mu x^i)(\frac{-i}{\kappa}\partial_0 x^0) + ... = x^0 x^i - \frac{i}{\kappa} x^i + ...$$

Zbog toga što je $\mathcal{F}_2^{-1} \sim P_0 \sim \partial_0$ sve prostornolike funkcije komutiraju.

Deformirana algebra diferencijalnih formi $\Omega^{\mathcal{F}}$

$(\Omega, \wedge, *, \mathrm{d})$ pri čemu $(\Omega, \wedge)$ čini UA algebru dok su d i $*$ lin. operatori koji se mogu definirati djelovanjem na r-formu:

$$\omega = \frac{1}{r!}\omega_{\mu_1\mu_2\ldots\mu_r}\mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \ldots \wedge \mathrm{d}x^{\mu_r} \in \Omega_r$$

$$\mathrm{d}\omega = \frac{1}{r!}\partial_\nu\omega_{\mu_1\mu_2\ldots\mu_r}\mathrm{d}x^{\nu} \wedge \mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \ldots \wedge \mathrm{d}x^{\mu_r} \in \Omega_{r+1}$$

$$*\omega = \frac{\sqrt{|g|}}{r!(m-r)!}\omega_{\mu_1\mu_2\ldots\mu_r}\epsilon^{\mu_1\mu_2\ldots\mu_r}{}_{\nu_{r+1}\ldots\nu_m}\mathrm{d}x^{\nu_{r+1}} \wedge \ldots \wedge \mathrm{d}x^{\nu_m} \in \Omega_{m-r}$$

Deformacija se provodi ranije definiranim postupkom čime se dobiva deformirana algebra diferencijalnih formi $(\Omega^{\mathcal{F}}, \wedge_\star, *^{\mathcal{F}}, \mathrm{d})$
Podalgebra 0-formi istovjetna algebri funkcija na mnogostrukosti

Valna jednadžba

$$\Box = (\mathrm{d} + *\mathrm{d}*)^2 \longrightarrow \Box^{\mathcal{F}} = (\mathrm{d} + *^{\mathcal{F}}\mathrm{d}*^{\mathcal{F}})^2$$

$$\Box^{\mathcal{F}}\phi = 0$$

$$\partial_\nu[\sqrt{|g|}g^{\mu\nu}] \star (1 - \frac{i}{\kappa}\partial_0)^3\partial_\mu\phi + \sqrt{|g|}g^{\mu\nu} \star \partial_\nu[(1 - \frac{i}{\kappa}\partial_0)^2\partial_\mu\phi] = 0$$

Također uobičajeno se dodaju članovi koji predstavljaju vezanje sa gravitacijskim poljem tipa $\frac{1}{6}R\phi$, ali ne daju značajan doprinos

Sadržaj

FLRW metrika

Metrika koja je svojstvena prostorvremenu sa homogenom raspodjelom materije - homogenost i izotropnost
Najjednostvaniji slučaj $g_{\mu\nu} = diag(-1, a(t)^2, a(t)^2, a(t)^2)$

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 \sim \frac{\ddot{a}}{a}$$
$$R = 6\left(\frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a}\right)$$

H - Hubbleov parametar današnje vrijednosti $\sim 10^{-18}\mathrm{s}^{-1}$
a - parametar skaliranja konvencionalno $a(t_0) = 1$
Aproksimacija $\omega >> H \sim 1/t_0$

Rješavanje jednadžbe na FLRW metrici

Nedeformirano rješenje

$$\phi = \frac{1}{a} \exp\left(ik\eta - i\mathbf{kx}\right) \qquad \eta = \int_0^{x^0} \frac{\mathrm{d}x'}{a(x')}$$

Ansatz

$$\phi = \frac{1}{a} \exp\left(ik\eta + \frac{i}{\kappa} F(t) - i\mathbf{kx}\right) = \frac{1}{a} \exp\left(if_{\mathbf{k}}(x^0) - i\mathbf{kx}\right)$$

dobiva se u prvom redu računa:

$$f_k(x^0) = \int_0^{x^0} \frac{k}{a} \left(1 - \frac{1}{\kappa} \frac{\dot{a}}{a^2} kx'\right) \mathrm{d}x'$$

Fizikalne posljedice

Disperzijske relacije

$$\omega_k = \frac{\partial f_k}{\partial x^0} = \frac{k}{a}\left(1 - \frac{1}{\kappa}\frac{\dot{a}}{a^2}kx^0\right)$$

Brzina valnog paketa

$$\mathbf{v} = \nabla_{\mathbf{k}}\omega_{\mathbf{k}} = \partial_0 \nabla_{\mathbf{k}} f_{\mathbf{k}}(x^0)$$

$$v = \frac{1}{a}\left(1 - \frac{1}{\kappa}\frac{2kx^0\dot{a}}{a^2}\right)$$

Razlika u vremenu dolaska fotona različith energija

Put koji foton prijeđe $s = \int_{t_e}^{t_a} v_k(x^0)\mathrm{d}x^0$

$$\int_{t_e}^{t_a} \frac{\mathrm{d}x}{a} = \int_{t_e}^{t_a+\Delta t} \frac{1}{a}\left(1 - \frac{1}{\kappa}\frac{2kx\dot{a}}{a^2}\right)\mathrm{d}x$$

$$\Delta t = \frac{2ka(t_a)}{\kappa}\int_{t_e}^{t_a} \frac{x\dot{a}}{a^3}\mathrm{d}x = \{a(t_a) = 1\} = 2\frac{k}{\kappa}\int_{t_e}^{t_a} \frac{x\dot{a}}{a^3}\mathrm{d}x$$

Reskaliranje integranda:

$$\int_{t_e}^{t_a} \frac{x\dot{a}}{a^3}\mathrm{d}x \sim t_0 \int_0^1 \frac{x\dot{a}(t_0 x)}{a(t_0 x)^3}\mathrm{d}x \sim t_0$$

$$\ln\frac{\Delta t}{s} \sim \ln\frac{k}{\kappa} + \ln\frac{t_0}{s} \sim -5$$