Higher-form and Higher-group Global Symmetries

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Abstract

We introduce the generalization of global symmetries through familiar ordinary symmetries. Higher-form symmetries are symmetries that lead to higher-form conserved currents. They are discussed with an example of free Maxwell theory with duality. 2-group symmetries, i.e. symmetries that allow for the mixing of background gauge fields under their respective gauge transformations are presented. We study the simplest abelian case and show how it is derived from ordinary product flavor symmetry by gauging.

1 Introduction

1.1 Context and Motivation

A symmetry is the property of a physical system that is preserved under some transformations. A family of such transformations can be described using groups - Lie groups for continuous symmetries and finite groups for discrete symmetries. Continuous and discrete symmetries correspond to continuous and discrete transformations, respectively. Amongst many other divisions of symmetries, we should mention the difference between external and internal symmetries where external refers to the symmetries of space-time, and internal symmetries correspond to the internal degrees of freedom of the theory. However, for our further observations, it will be most important to distinguish local from global symmetries. Global symmetries keep a property invariant for a transformation that is applied simultaneously at all points of space-time, whereas local symmetries are features invariant to transformations parametrized by space-time coordinates. Local symmetries are the foundation of gauge field theories, i.e. gauge theory is presented with a Lagrangian density $\mathcal{L}$ invariant to a smooth family of operations. Because gauge fields (which take values in the Lie algebra of the gauge group) are included in the Lagrangian density $\mathcal{L}$ to ensure its gauge invariance, gauge theories have additional, i.e. redundant degrees of freedom. For example, the photon has two physical polarizations, but the gauge field that we use to describe it in a relativistic manner has four components. The Standard Model, one of the most successful and accurate physical theories, is based on gauge symmetries.

Symmetries are of great importance in physics mostly due to Noether's theorem (which will be discussed later) that shows how, for every continuous global symmetry, there is a corresponding conservation law. Throughout classical mechanics, spatial and temporal invariances were known and used, as well as global spacetime symmetries for electrodynamics that were derived before Einstein's special theory of relativity. Nevertheless, the latter represents a new approach to the application of symmetry in physics since, unlike those before him, Einstein derived the laws from the symmetries. The significance of symmetries in physics was quickly made clear in quantum mechanics where the application of the theory of groups and their representations played a crucial role.

The need to develop a universal tool for the application of symmetries became noticeable in quantum field theory as the study of higher-form gauge fields became standard in mathematics and physics. Roughly speaking, generalizing global symmetries is applying the concept to objects of higher dimensions. Such generalized global symmetries [2] have shown to have applications within string theory and condensed matter physics, as well as in the study of extended operators and defects and of the anomaly structure in quantum field theory. They have been a subject of discussion in various fields of theoretical physics recently as they provide a new and organized language to think about symmetry principles.

1.2 Mathematical Introduction

To be able to understand the formalisms in the following sections, some definitions might come as useful reminders.

A q-form (differential form) is a totally antisymmetric $(0, q)$ tensor field. For a more intuitive approach, a 0-form is a function and a 1-form is a covector. Generally, the antisymmetry of q-forms has the following consequence: there can't be any form of degree higher than the dimension n of the manifold on which the form is defined. Note that an n-form is often referred to as *top form*, or *volume form*.

Let ω be a q-form and η a p-form. The *exterior product* or *wedge product* is a construction of a $(p+q)$-tensor:

$$(\omega \wedge \eta)_{\mu_1 \ldots \mu_q \nu_1 \ldots \nu_p} = \frac{(q+p)!}{p!q!} \omega_{[\mu_1 \ldots \mu_q} \eta_{\nu_1 \ldots \nu_p]} \tag{1}$$

via the tensor product that is antisymmetrized to ensure some properties, such as $\omega \wedge \omega = 0$ if ω is an odd-degree form. It can also be shown that:

$$\omega \wedge \eta = (-1)^{qp} \eta \wedge \omega \ . \tag{2}$$

To build some intuition, we can take a look at a special case where $p = q = 1$, meaning we take ω and η to both be 1-forms $\omega^{(1)}$ and $\eta^{(1)}$. It is easy to show what their exterior product is in terms of the tensor product $\otimes$:

$$\omega^{(1)} \wedge \eta^{(1)} = \omega^{(1)} \otimes \eta^{(1)} - \eta^{(1)} \otimes \omega^{(1)} \ .$$

The *exterior derivative* (is a map that) in local coordinates, acts as:

$$\mathrm{d}\omega = \frac{1}{q!} \frac{\partial \omega_{\mu_1 \ldots \mu_q}}{\partial x^\nu} dx^\nu \wedge dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_q} \ , \tag{3}$$

where ω is a q-form. The exterior derivative defined by (3) returns a $(q+1)$-form. We can think of the exterior derivative as an antisymmetric covariant derivative.

It is important to note that, due to the antisymmetry, if we act on equation (3) with the exterior derivative again, we obtain:

$$\mathrm{d}\left(\mathrm{d}\omega\right) = 0 \ ,$$

which is true for every q-form and often written in the following form.

$$\mathrm{d}^2 = 0 \tag{4}$$

The exterior derivative of a wedge product for ω and η of degrees as given before, is:

$$\mathrm{d}\left(\omega \wedge \eta\right) = \left(\mathrm{d}\omega\right) \wedge \eta + \left(-1\right)^q \omega \wedge \left(\mathrm{d}\eta\right) \quad . \tag{5}$$

The *Hodge dual* is a map that takes a q-form ω to an $(n-q)$-form, denoted $*\omega$ as follows.

$$(*\omega)_{\mu_1 \ldots \mu_{n-q}} = \frac{1}{q!}\sqrt{|g|}\epsilon_{\mu_1 \ldots \mu_{n-q}\nu_1 \ldots \nu_q}\omega^{\nu_1 \ldots \nu_q} \tag{6}$$

Note that the Hodge dual is independent of the choice of coordinates. Using the Hodge dual, we can define an inner product of two q-forms ω and η:

$$\langle \eta, \omega \rangle = \int_M \eta \wedge *\omega \ , \tag{7}$$

where it can be seen that the dimension of $\eta \wedge *\omega$ is equal to the dimension n of the manifold M. To get a better picture of the Hodge star operator, we should keep in mind that it gives the part of the manifold orthogonal to the differential form that it is acting on. A common 3D example follows:

$$*dx = dy \wedge dz \ .$$

Now that all of the important definitions are given, we can revisit some familiar notions of global symmetries.

2 Ordinary Global symmetries

2.1 Noether's Theorem

As said before, Noether's theorem shows that for every continuous global symmetry, there is a corresponding *conserved current* j^μ given with:

$$\partial_\mu j^\mu = 0 \ , \tag{8}$$

or more generally, using the previously introduced language:

$$*\mathrm{d} * j^{(1)} = 0 \ , \tag{9}$$

where $j^{(1)}$ is a 1-form. If we act on the equation (9) with the Hodge dual once again, we get the final expression:

$$\mathrm{d} * j^{(1)} = 0 \ . \tag{10}$$

For the observation in this section, we will use the first expression (8).

The *conserved charge* is defined as

$$Q = \int j^0 d^3x \ , \tag{11}$$

or, once again, more generally:

$$Q = \int *j^{(1)} \ . \tag{12}$$

As with the conserved current, we will use the first of the two expressions, equation (11), and show Noether's theorem in the language of quantum field theory.

Let's take a look at an action S:

$$S = \int \mathcal{L}(\psi, \partial_\mu \psi, x) d^4x \quad , \tag{13}$$

and a transformation:

$$\psi \to \psi + \alpha \Delta \psi \ . \tag{14}$$

In the equations above, ψ is a field (let's say a fermionic field that we will later couple to the electromagnetic background field), α is an infinitesimal parameter and all of the other notations are standard. The Lagrangian density $\mathcal{L}$ also transforms as shown below.

$$\mathcal{L} \to \mathcal{L} + \alpha \partial_\mu \mathcal{J}^\mu \tag{15}$$

Note that $\mathcal{J}^\mu$ is not the conserved current. Variation of the Lagrangian density $\delta\mathcal{L}$ is, of course, equal to the second term in (15) which gives us the following equation.

$$\begin{aligned} \alpha \partial_\mu \mathcal{J}^\mu &= \delta \mathcal{L} \\ &= \frac{\partial \mathcal{L}}{\partial \psi} \delta \psi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \delta(\partial_\mu \psi) \end{aligned}$$

Including $\delta\psi = \alpha\Delta\psi$ and regrouping terms results with:

$$\alpha \partial_\mu \mathcal{J}^\mu = \alpha \left[\frac{\partial \mathcal{L}}{\partial \psi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \right) \right] \Delta \psi + \alpha \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \Delta \psi \ ,$$

where we recognize the LHS of the Euler-Lagrange equation in square brackets, meaning we can replace it with zero. We finally get the expression:

$$\alpha \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \Delta \psi - \mathcal{J}^\mu \right) = 0 \ . \tag{16}$$

By comparison with (8), expression in the brackets in equation (16) is the conserved current j^μ:

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \Delta \psi - \mathcal{J}^\mu \ . \tag{17}$$

We will now show that the charge defined with (11) is conserved, using the equation of continuity (8) in the third step.

$$\begin{aligned}\frac{dQ}{dt} &= \frac{d}{dt}\int j^0 d^3x \\ &= \int \partial_0 j^0 d^3x \\ &= -\int \partial_i j^i d^3x \\ &= -\int_{\mathbb{R}^3} \nabla \cdot \vec{j} d^3x\end{aligned}$$

Finally, by applying Gauss' theorem,

$$\frac{dQ}{dt} = 0 \tag{18}$$

we obtain equation (18) which means that the charge is conserved.

2.2 Abelian Global Symmetries and the Electric Charge

To be able to understand the generalization of global symmetries, let's first revisit a familiar global symmetry of the Standard Model. There are multiple laws of conservation within the Standard Model that come from global symmetries. To see how conserved charges arise, we will show the example of the electric charge.

Let's apply the discussion shown in section 2.1 to a free fermionic field ψ coupled to the electromagnetic field, as shown with action S:

$$S = \int \overline{\psi} \left(i\gamma^\mu D_\mu - m\right) \psi d^4x \ , \tag{19}$$

where

$$D_\mu = \partial_\mu + ieA_\mu \ , \tag{20}$$

with standard notation. Note that the dynamical term is not of interest here, hence we are using A simply as a background gauge field. We will show that the Lagrangian density is invariant for a phase transformation of ψ:

$$\psi \to e^{i\alpha}\psi \tag{21}$$

which is characterized by $U(1)$ group of transformations. Expanding the transformation using the Taylor series to the first order, and comparing with (14) we get $\Delta\psi = i\psi$ and $\Delta\overline{\psi} = -i\overline{\psi}$. One should note that the two behave differently only with the respect to the sign. Due to the latter, when deriving the change in the Lagrangian density:

$$\Delta\mathcal{L} = \Delta\overline{\psi} \left(i\gamma^\mu D_\mu - m\right) \psi + \overline{\psi} \left(i\gamma^\mu D_\mu - m\right) \Delta\psi \tag{22}$$

the terms cancel and we get zero.

$$\Delta\mathcal{L} = 0 \tag{23}$$

The Lagrangian density is truly invariant to such transformations. It immediately follows from (15):

$$\mathcal{J}^\mu = 0 \ . \tag{24}$$

Let's calculate the conserved current by plugging what we have obtained so far in equation (17).

$$\begin{aligned} j^\mu &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \Delta\psi - \mathcal{J}^\mu \\ &= i\overline{\psi}\gamma^\mu \Delta\psi \end{aligned}$$

Altogether, the conserved current is given with:

$$j^\mu = -\overline{\psi}\gamma^\mu \psi \ . \tag{25}$$

The result for the conserved charge depends on ψ and therefore cannot be calculated here.

2.3 Comments on Anomalies

Upon attempting to quantize a theory with a global symmetry, an *anomaly* can occur. For the following sections, an understanding of anomalies in quantum field theory will be necessary. Roughly speaking, an anomaly is a classical symmetry that does not remain when theory is quantized. Some anomalies can be canceled by adding terms to the action. The most important to us will be 't Hooft anomalies which present an obstruction to gauging a global symmetry. A global symmetry with a 't Hooft anomaly remains a symmetry in the quantum theory, but when the symmetry is coupled to a background gauge field, the charges that were previously conserved are then not.

An anomaly $\mathcal{A}(\mathcal{G})$ is a term within the (effective [1]) action that shifts the action and corresponds to a non-conservation law. Here, we have used notation $\mathcal{G}$ for the background gauge fields. An anomaly is usually summarized by a $(d+2)$-form gauge invariant anomaly polynomial $\mathcal{I}^{(d+2)}(\mathcal{G})$, meaning that the background gauge fields and their gauge transformations $\mathcal{G}$ are extended to $d+2$ dimensions. The relation between $\mathcal{A}(\mathcal{G})$ and $\mathcal{I}^{(d+2)}(\mathcal{G})$, as well as the relation between the polynomials that present a procedure for the extension of $\mathcal{A}(\mathcal{G})$ to $\mathcal{I}^{(d+2)}(\mathcal{G})$ are given as:

$$\mathcal{A}(\mathcal{G}) = 2\pi i \int_{M_d} \mathcal{I}^{(d)}(\mathcal{G}, \delta\mathcal{G}) \ , \tag{26}$$

$$\mathrm{d}\mathcal{I}^{(d)}(\mathcal{G}, \delta\mathcal{G}) = \delta\mathcal{I}^{(d+1)}(\mathcal{G}) \ , \tag{27}$$

$$\mathrm{d}\mathcal{I}^{(d+1)}(\mathcal{G}) = \mathcal{I}^{(d+2)}(\mathcal{G}) \tag{28}$$

Note that the procedure can be used in both directions.

Furthermore, a notion of *Wilson loops* and *'t Hooft loops* will be made, so let's take a look at their definitions. The Wilson line U is an object that tells us how a complex vector carried by a particle moves around the manifold with connection A (a Lie-algebra valued gauge field).

$$U = \mathcal{P} \exp\left(i \int_{x_i}^{x_f} A \right)$$

[1] Anomalies are recognized in action S, but $\mathcal{A}(\mathcal{G})$ refers to the shift in the effective action $W(\mathcal{G}) = -\log Z(\mathcal{G})$, where $Z(\mathcal{G})$ is the partition function.

Here, $\mathcal{P}$ stands for the path ordering, while x_i and x_f are the initial and final points of the particle's movement, respectively. In mathematics, this notion is called holonomy. The *Wilson loop* $W(L)$ is a gauge invariant object, an observable, defined as the trace of the Wilson line on a closed path L.

$$W(L) = \mathrm{tr}\left[\mathcal{P}\ i \int_L A\right] \tag{29}$$

The *'t Hooft loop* $H(L)$ is also an observable, similar to the Wilson loop, and related to it as shown below.

$$H(L_1)W(L_2) = Z^{l(L_1,L_2)}W(L_2)H(L_1) \tag{30}$$

In the expression (30), Z stands for an element in the center of the gauge group, and $l(L_1, L_2)$ is the Gaussian linking number between the two spatial loops. Since they are observables, these objects are of great importance, particularly in non-abelian theories (such as Yang-Mills), where electric and magnetic fields are not observables.

3 Generalized global symmetries

To start with, a *q- form global symmetry* is a global symmetry for which the conserved current is a $(q+1)$-form and the conserved charges are of dimension q. In this language, 0-form global symmetries are the ordinary global symmetries that were previously described. Many of the properties of 0-form global symmetries can be applied. These generalized global symmetries are not some exotic generalizations in complicated theories, but rather appear naturally in gauge theories. We should emphasize that, if there are no 't Hooft anomalies, the theory can be gauged. To present generalized global symmetries, we will not rely on a specific Lagrangian density but rather characterize the charged objects as abstract operators, making the layout general.

If we introduce a $U(1)^{(q)}_B$ symmetry that arises from a conserved $(q+1)$-form current $j^{(q+1)}_B$ that satisfies

$$\mathrm{d} * j^{(q+1)}_B = 0\ , \tag{31}$$

the conserved charges are given with:

$$Q = \int *j^{(q+1)} \tag{32}$$

The charged objects for these symmetries are q-dimensional. For instance, in the simplest case of $q = 1$, the charged objects can be line operators, such as the Wilson and 't Hooft lines mentioned earlier. This justifies that these symmetries are not something strange but are essentially present in any theory that has extended observables like Wilson loops.

The classical source for the current is an abelian $(q+1)$-form gauge field $B^{(q+1)}$. The action must contain the following term:

$$\int B^{(q+1)} \wedge *j^{(q+1)}_B\ . \tag{33}$$

Under $U(1)^{(q)}_B$ transformation, the gauge field should transform as follows:

$$B^{(q+1)} \to B^{(q+1)} + \mathrm{d}\lambda^{(q)}_B\ , \tag{34}$$

where $\lambda^{(q)}_B$ is a q-form gauge parameter. The next step is to look at an example of free Maxwell theory.

3.1 Free Maxwell Theory

Let us consider a free $U(1)_c^{(0)}$ gauge theory with gauge field $c^{(1)}$ and the corresponding field strength $f_c^{(2)} = \mathrm{d}c^{(1)}$ and two 1-form global symmetries: "electric" $U(1)_e^{(1)}$ and "magnetic" $U(1)_m^{(1)}$ with respective background fields $B_m^{(2)}$ and $B_e^{(2)}$.

$$U(1)_e^{(1)} \times U(1)_m^{(1)} \tag{35}$$

For currents defined as:

$$j_e^{(2)} = -\frac{1}{e^2} f_c^{(2)} \tag{36}$$

$$j_m^{(2)} = -\frac{i}{2\pi} * f_c^{(2)} \ , \tag{37}$$

we obtain corresponding conservation laws if we use source-free Maxwell equations, written below.

$$\mathrm{d} * f_c^{(2)} = \mathrm{d} f_c^{(2)} = 0 \tag{38}$$

The source-free Maxwell equations given with (38), of course, relate to the familiar layout of Maxwell's equations: $\mathrm{d} * f_c^{(2)} = 0$ is associated with source-free Gauss' law and source-free Ampère's law, whereas $\mathrm{d} f_c^{(2)} = 0$ corresponds to Gauss' law for magnetism and Faraday's law. If we apply the exterior derivative to (36) and (37), and plug Maxwell's equations in, it is easy to see that currents $j_e^{(2)}$ and $j_m^{(2)}$ are conserved. So, we are building a theory of a dynamical gauge field $c^{(1)}$ in the environment of the two background (non-dynamical) gauge fields who, due to their $U(1)_{e,m}^{(1)}$ symmetries undergo gauge transformations given with:

$$B_{e,m}^{(2)} \to B_{e,m}^{(2)} + \mathrm{d}\Lambda_{e,m}^{(1)} \ . \tag{39}$$

To be able to write the action S for this theory, i.e. to couple the dynamical field to the two background gauge fields, two approaches can be taken: "electric" and "magnetic". Surely, the two approaches must be equivalent, meaning that they show the duality of the theory which might be useful. First, we will take the "electric" approach: the gauge field $c^{(1)}$ shifts under $U(1)_e^{(1)}$ background transformation, but remains the same under $U(1)_m^{(1)}$ background transformation. Since $f_c^{(2)} = \mathrm{d}c^{(1)}$, the same is true for the field strength f_c.

$$c^{(1)} \to c^{(1)} + \Lambda_e^{(1)} \ , \quad f_c^{(2)} \to \mathrm{d}\Lambda_e^{(1)} \tag{40}$$

If we couple the gauge field $c^{(1)}$ to the background gauge fields, we expect the action S to have typical terms $\int \left(B_e^{(2)} \wedge *J_e^{(2)} + B_m \wedge *J_m^{(2)} \right)$, which translates to:

$$\begin{aligned} S\left(B_e^{(2)}, B_m^{(2)}, c^{(1)}\right) =& \frac{1}{2e^2} \int \left(f_c^{(2)} - B_e^{(2)}\right) \wedge * \left(f_c^{(2)} - B_e^{(2)}\right) \\ &+ \frac{i}{2\pi} \int B_m^{(2)} \wedge f_c^{(2)} \quad . \end{aligned} \tag{41}$$

In the equation above, we have also used the fact that we want the theory to be invariant under electric background transformation. The first term ensures this, since $\mathrm{d}\Lambda_e$ is added to both $f_c^{(2)}$ and

$B_e^{(2)}$ after the background transformation. It is easy to check that, after the transformation, $\mathrm{d}\Lambda_e$ derived from the transformation of $f_c^{(2)}$ and $\mathrm{d}\Lambda_e$ obtained from the transformation of $B_e^{(2)}$, cancel each other out, as they come with opposite signs. The first term is often referred to as kinetic, and the second term is referred to as magnetic. As neither $f_c^{(2)}$ nor $B_e^{(2)}$ shift under a $U(1)_m^{(1)}$ background transformation, the first term is also invariant to magnetic background transformations. What remains is to ensure the invariance of the second term. The second term in (41) is invariant under magnetic background transformations, which can be easily verified using source-free Maxwell equations (38), expression (5) and Stokes' theorem.

$$\begin{aligned}\int \left(B_m^{(2)} + \mathrm{d}\Lambda_m^{(1)}\right) \wedge f_c^{(2)} &= \int B_m^{(2)} \wedge f_c^{(2)} + \int \mathrm{d}\Lambda_m^{(1)} \wedge f_c \\ &= \int B_m^{(2)} \wedge f_c^{(2)} + \int \mathrm{d}\left(\Lambda_m^{(1)} \wedge f_c\right) - \int \Lambda_m^{(1)} \wedge \mathrm{d} \wedge f_c^{(2)} \\ &= \int B_m^{(2)} f_c^{(2)}\end{aligned}$$

However, this (magnetic) term experiences a shift under $U(1)_e^{(1)}$ background transformation - let's once again use equation (5) and Stokes' theorem to see exactly how.

$$\begin{aligned}\int B_m^{(2)} \wedge f_c^{(2)} \rightarrow &\int B_m^{(2)} \wedge \left(f_c^{(2)} + \mathrm{d}\Lambda_e^{(1)}\right) \\ &= \int B_m^{(2)} \wedge f_c^{(2)} + \int B_m^{(2)} \wedge \mathrm{d}\Lambda_e^{(1)} \\ &= \int B_m^{(2)} \wedge f_c^{(2)} - \int \mathrm{d}\left(B_m^{(2)} \wedge \Lambda_e^{(1)}\right) + \int \mathrm{d}\Lambda_e^{(1)} \wedge B_m^{(2)} \\ &= \int B_m^{(2)} \wedge f_c^{(2)} + \int \mathrm{d}\Lambda_e^{(1)} \wedge B_m^{(2)}\end{aligned}$$

The term given with $\int \mathrm{d}\Lambda_e^{(1)} \wedge B_m^{(2)}$ constitutes a 't Hooft anomaly between $U(1)_e^{(1)}$ and $U(1)_m^{(1)}$. As previously announced, anomalies are usually shown using a $(d+2)$-form, i.e. a 6-form polynomial $\mathcal{I}^{(6)}$. Using the procedure explained in $(26) - (28)$, as well as, once again expression (5) and Stokes' theorem as before, we obtain the following expression.

$$\mathcal{I}^{(6)} \supset \frac{1}{4\pi^2} dB_e^{(2)} \wedge \mathrm{d}B_m^{(2)} \tag{42}$$

To show the "electric-magnetic" duality of the theory, we will derive the dual "magnetic" representation starting from the "electric" presentation of the theory. We should mention that 't Hooft anomalies are reproduced in any possible description of the theory, meaning that the magnetic formulation of free Maxwell theory should reproduce the same 't Hooft anomaly as (42). Although gauge symmetries may differ in the dual descriptions, the global symmetries of the theory must be the same in the dual formulations. This is an important fact to remember about any kind of duality in physics: the global symmetries must always match, even if the gauge symmetries might not.

The dualization is done by considering an extended theory with action $\tilde{S}$ that includes a Lagrange multiplier $\tilde{c}^{(1)}$ which is also a 1-form gauge field associated with its own $U(1)_{\tilde{c}}^{(1)}$ gauge symmetry.

$$\tilde{S}\left(B_e^{(2)}, B_m^{(2)}, c^{(1)}, \tilde{c}^{(1)}\right) = S\left(B_e^{(2)}, B_m^{(2)}, c^{(1)}\right) - \frac{i}{2\pi}\int \mathrm{d}\tilde{c}^{(1)} \wedge f_c^{(2)} \tag{43}$$

Note that Bianchi's identity for $f_c^{(2)}$ is still satisfied. The appropriate shift of $\tilde{c}^{(1)}$ under $U(1)_m^{(1)}$ background gauge transformations:

$$\tilde{c}^{(1)} \to \tilde{c}^{(1)} + \Lambda_m^{(1)} \ , \tag{44}$$

ensures the invariance under background gauge transformations up to the 't Hooft anomaly obtained earlier. We now want to find the appropriate equation of motion for $f_c^{(2)}$ for action $\tilde{S}$ to depend only on $B_e^{(2)}$, $B_m^{(2)}$ and the new gauge field $\tilde{c}^{(1)}$. In other words, we want to "lose" the dependence of action on $c^{(1)}$ and replace it with dependence on $\tilde{c}^{(2)}$. The equation of motion for $f_c^{(2)}$ is obtained by varying $\tilde{S}$ over $f_c^{(2)}$, as shown in the next equation.

$$\begin{aligned}\delta\tilde{S} = \int \Bigg[&\frac{1}{2e^2}\delta f_c^{(2)} \wedge * \left(f_c^{(2)} - B_e^{(2)}\right) + \frac{1}{2e^2}\left(f_c^{(2)} - B_e^{(2)}\right) \wedge *\delta f_c^{(2)} \\ &+ \frac{i}{2\pi}\left(B_m^{(2)} - \mathrm{d}\tilde{c}^{(1)}\right) \wedge \delta f\Bigg]\end{aligned} \tag{45}$$

Due to symmetry of $\wedge *$, the second term can be written as $\frac{1}{2e^2}\delta f_c^{(2)} \wedge * \left(f_c^{(2)} - B_e^{(2)}\right)$, and due to graded commutativity, the last term can be replaced with
$\frac{i}{2\pi}\delta f \wedge \left(B_m^{(2)} - \mathrm{d}\tilde{c}^{(1)}\right)$.
The latter means that δf can be extracted from all of the terms as follows:

$$\delta\tilde{S} = \int \delta f_c^{(2)} \wedge \left[\frac{1}{e^2} * \left(f_c^{(2)} - B_e^{(2)}\right) + \frac{i}{2\pi}\left(B_m^{(2)} - \mathrm{d}\tilde{c}^{(1)}\right)\right] \ , \tag{46}$$

and the equation of motion for $f_c^{(2)}$ is obtained when the expression in the square brackets is set to zero.

$$* \left(f_c^{(2)} - B_e^{(2)}\right) = \frac{ie^2}{2\pi}\left(\mathrm{d}\tilde{c}^{(1)} - B_m^{(2)}\right) \tag{47}$$

As $f_c^{(2)}$ is now easily expressed from the previous equation (47), that expression can be included in (43). The dual presentation of the theory with action $\tilde{S}$, now depending only on $B_e^{(2)}$, $B_m^{(2)}$ and $\tilde{c}^{(1)}$, is, therefore, derived.

$$\begin{aligned}\tilde{S}\left(B_e^{(2)}, B_m^{(2)}, \tilde{c}^{(1)}\right) =& \frac{e^2}{8\pi}\int \left(\mathrm{d}\tilde{c}^{(1)} - B_m^{(2)}\right) \wedge * \left(\mathrm{d}\tilde{c}^{(1)} - B_m^{(2)}\right) \\ &- \frac{i}{2\pi}\int B_e^{(2)} \wedge \left(\mathrm{d}\tilde{c} - B_m^{(2)}\right)\end{aligned} \tag{48}$$

This expression shows that the duality generates a counterterm proportional to $\int B_e^{(2)} \wedge B_m^{(2)}$ which will reproduce the same 't Hooft anomaly as before with (42). The conserved currents within this theory are given with the following equations and related to the currents defined in the "electric" presentation as shown here.

$$\tilde{J_m}^{(2)} = \frac{i}{2\pi} * \mathrm{d}\tilde{c}^{(1)} = -J_e^{(2)} \ , \ \tilde{J}_e^{(2)} = -\frac{e^2}{4\pi^2}\mathrm{d}\tilde{c}^{(1)} = J_m^{(2)} \tag{49}$$

Holonomies of $c^{(1)}$ and $\tilde{c}^{(1)}$ around a closed 1-cycle L are Wilson loops $W_m(L)$ and 't Hooft loops $H_n(L)$, defined with (29) and (30), given as:

$$W_m = \exp\left(im\int_L c^{(1)}\right), \quad H_n(L) = \exp\left(in\int_L \tilde{c}^{(1)}\right) \ , \tag{50}$$

where $m, n \in \mathbb{Z}$ are charges of the Wilson and 't Hooft loops respectively. When moving from one formulation to another, we exchange $c^{(1)} \leftrightarrow \tilde{c}^{(1)}$, and so do the loops $W_m(L) \leftrightarrow H_m(L)$.

4 2-group symmetries

A quantum field theory has a 2-group symmetry, according to [2], if it can be coupled to a 2-form background gauge field (let's denote it $B^{(2)}$) that undergoes a 2-group shift in addition to its own $U(1)_B^{(1)}$ background gauge transformations. In other words, we are taking a look at global symmetries where the mixing of background gauge fields under their respective gauge transformations is allowed. Here, we do not explore 2-groups themselves but rather discuss 2-group background gauge fields. To keep it straightforward, we will look at the simplest example where the mixing of a background gauge field $A^{(1)}$ for a 0-form flavor symmetry, i.e. $U(1)_A^{(0)}$ and a 2-form background gauge field $B^{(2)}$ for previously mentioned 1-form symmetry is involved.
Such 2-group symmetry is said to be abelian and denoted as shown below.

$$U(1)_A^{(0)} \times_{\hat{\kappa_A}} U(1)_B^{(1)} \tag{51}$$

Here, $\hat{\kappa_A} \in \mathbb{Z}$ [1]is a 2-group structure constant that characterizes the 2-group symmetry. To see what $\hat{\kappa_A}$ means, we should take a look at transformation rules for our gauge fields $A^{(1)}$ and $B^{(2)}$. The transformation rule for $A^{(1)}$ remains standard:

$$A^{(1)} \to A^{(1)} + \mathrm{d}\Lambda_A^{(0)} \ , \tag{52}$$

but, as said before, $B^{(2)}$ undergoes an additional shift.

$$B^{(2)} \to B^{(2)} + \mathrm{d}\Lambda_B^{(1)} + \frac{\hat{\kappa_A}}{2\pi}\Lambda_A^{(0)} F_A^{(2)} \tag{53}$$

In the previous expression, $F_A^{(2)} = \mathrm{d}A^{(1)}$ is the field strength. The consistency of the transformation rule given with (53) is ensured with $\hat{\kappa_A}$ being quantized. We should mention that $\hat{\kappa_A}$ characterizes the 2-group symmetry because it doesn't change with the rescaling of the gauge fields. We see that in equation (53) there is an additional shift proportional to the field strength $F_A^{(2)}$, meaning that we cannot non-trivially set gauge field $A^{(1)}$ without it affecting $B^{(2)}$. We should recognize that for $\hat{\kappa_A} = 0$, the 2-group shift in (53) disappears. Therefore, the 2-group symmetry dissolves into product symmetry.

$$U(1)_A^{(0)} \times U(1)_B^{(1)}$$

Many quantum field theories posses the 2-group symmetry given with (51) such as QED with many flavours [1].

It can be shown that the 2-group symmetry described with (51) arises from a "parent" theory with

$$U(1)_A^{(0)} \times U(1)_C^{(0)} \tag{54}$$

flavor symmetry, with $C^{(1)}$ being the corresponding background gauge field. Because of this, it will be useful to take a look at parent theories with such abelian 0-form flavor symmetry. We will show how gauging $U(1)_C^{(0)}$ leads to a new 1-form global symmetry. Let's start by gauging $U(1)_C^{(0)}$ from (54) by promoting $C^{(1)}$ and its field strength $F_C^{(2)}$ to dynamical fields. We will denote the change using $C \to c$.

$$U(1)_C^{(0)} \to U(1)_c^{(0)},\ C^{(1)} \to c^{(1)},\ F_C^{(2)} \to f_c^{(2)}$$

The action should contain:

$$S \supset \frac{1}{2e^2}\int f_c^{(2)} \wedge * f_c^{(2)} + \frac{i\theta}{8\pi^2}\int f_c^{(2)} \wedge f_c^{(2)}\ . \tag{55}$$

In the previous expression, we have added the theta-term[2].

Now, we want to check for anomalies to make sure $U(1)_C^{(0)}$ can be gauged. The most general anomaly 6-form polynomial, constructed of the field strengths $F_A^{(2)}$ and $F_C^{(2)}$ is:

$$\begin{aligned}\mathcal{I}^{(6)} =& \frac{1}{(2\pi)^3}\Big[\frac{\kappa_{A^3}}{3!}F_A^{(2)} \wedge F_A^{(2)} \wedge F_A^{(2)} + \frac{\kappa_{A^2C}}{2!}F_A^{(2)} \wedge F_A^{(2)} \wedge F_c^{(2)} \\ &+ \frac{\kappa_{AC^2}}{2!}F_A^{(2)} \wedge F_C^{(2)} \wedge F_C^{(2)} + \frac{\kappa_{C^3}}{3!}F_C^{(2)} \wedge F_C^{(2)} \wedge F_C^{(2)}\Big]\ .\end{aligned} \tag{56}$$

For further analysis, we need a bit more context. An anomaly polynomial $\mathcal{I}^{(d+2)}$ is called *reducible* if it can be written as a product of closed, gauge invariant polynomials $\mathcal{J}^{(p)}$ and $\mathcal{K}^{(d+2-p)}$ of lower degree.

$$\begin{aligned}\mathcal{I}^{(d+2)}_{reducible} &= \mathcal{J}^{(p)} \wedge \mathcal{K}^{(d+2-p)} \\ &= \mathrm{d}\mathcal{I}^{(d+1)}_{reducible}\end{aligned}$$

When trying to obtain $\mathcal{I}^{d+1}_{reducible}$, which is the procedure described earlier with (26) – (28), an ambiguity gets involved, since the exterior derivative can be removed from either factor in (56). This can be described using a real parameter s:

$$\mathcal{I}^{(d+1)}_{reducible} = \mathcal{J}^{(p-1)} \wedge \mathcal{K}^{(d+2-p)} + s\mathrm{d}\left(\mathcal{J}^{(p-1)} \wedge \mathcal{K}^{(d+1-p)}\right)\ , \tag{57}$$

where

$$\mathcal{J}^{(p)} = \mathrm{d}\mathcal{J}^{(p-1)}, \mathcal{K}^{(d+2-p)} = \mathrm{d}\mathcal{K}^{(d+1-p)}\ .$$

[2] the gauge invariant term that can be added to a 4-dimensional action, quadratic in field strength

Another ambiguity arises when a similar procedure is used further[1]. Altogether, for $d=4$ dimensions, we obtain:

$$\begin{aligned}\mathcal{I}^{(5)} = \frac{1}{(2\pi)^3} &\left[\frac{\kappa_{A^3}}{3!} A^{(1)} \wedge F_A^{(2)} \wedge F_A^{(2)} + \frac{\kappa_{A^2C}}{2!} A^{(1)} \wedge F_A^{(2)} \wedge F_c^{(2)}\right. \\ &+ \left.\frac{\kappa_{AC^2}}{2!} A^{(1)} \wedge F_C^{(2)} \wedge F_C^{(2)} + \frac{\kappa_{C^3}}{3!} C^{(1)} \wedge F_C^{(2)} \wedge F_C^{(2)}\right] \\ &+ s\mathrm{d}\left(A^{(1)} \wedge F_A^{(2)} \wedge C^{(1)}\right) + t\mathrm{d}\left(A^{(1)} \wedge C^{(1)} \wedge F_C^{(2)}\right)\end{aligned} \tag{58}$$

Using the procedure described in (26) − (28), we compute the expression for anomaly $\mathcal{A}_C$.

$$\mathcal{A}_C = \frac{i}{4\pi^2} \int_{M_4} \Lambda_C^{(0)} \left(\frac{\kappa_{C^3}}{3!} F_C^{(2)} \wedge F_C^{(2)} + sF_A^{(2)} \wedge F_A^{(2)} + tF_A^{(2)} \wedge F_C^{(2)}\right) \tag{59}$$

For $U(1)_C^{(0)}$ to be gauged, new $U(1)_c^{(0)}$ gauge transformations must be anomaly-free, i.e. we impose $\mathcal{A}_C = 0$. This is satisfied when:

$$\kappa_{C^3} = s = t = 0 \ . \tag{60}$$

This leads to anomaly that appears under $U(1)_A^{(0)}$ gauge transformations being of the form presented here:

$$\mathcal{A}_A = \frac{i}{4\pi^2} \int_{M_4} \Lambda_A^{(0)} \left(\frac{\kappa_{A^3}}{3!} F_A^{(2)} \wedge F_A^{(2)} + \frac{\kappa_{A^2C}}{2!} F_A^{(2)} \wedge F_C^{(2)} + \frac{\kappa_{AC^2}}{2!} F_C^{(2)} \wedge F_C^{(2)}\right) \ . \tag{61}$$

Consequently, the following non-conservation law is obtained:

$$\mathrm{d} * j_A^{(1)} = -\frac{i}{4\pi^2} \left(\frac{\kappa_{A^3}}{3!} F_A^{(2)} \wedge F_A^{(2)} + \frac{\kappa_{A^2C}}{2!} F_A^{(2)} \wedge F_C^{(2)} + \frac{\kappa_{AC^2}}{2!} F_C^{(2)} \wedge F_C^{(2)}\right) \ . \tag{62}$$

When the gauge field $C^{(1)}$ is promoted to the dynamical gauge field $c^{(1)}$, together with its field strength, these anomalous shifts become operator-valued and need to be accounted for.
First, we will examine the mixed κ_{AC^2} anomaly in the non-conservation law shown above.

$$\mathrm{d} * j_A^{(1)} \supset -\frac{i\kappa_{AC^2}}{8\pi^2} f_c^{(2)} \wedge f_c^{(2)} \tag{63}$$

The term shown in (63) violates the conservation of the current but can be accounted for if some transformation rules are changed. We should emphasize that this will not affect the dynamics of the theory and is different than cancellations of anomalies which involve coupling the theory to additional fields. The anomaly is resolved if θ in (55) is promoted to a background field that shifts under $U(1)_A^{(0)}$ background gauge transformations as:

$$\theta \to \theta - \kappa_{AC^2} \Lambda_A^{(0)} \ , \tag{64}$$

which shows that, since θ is not dynamical, but rather a background field, $U(1)_A^{(0)}$ is explicitly broken and there is no profile of θ that would stay the same after transformation (64). If we want our theory to not be explicitly broken, we must demand:

$$\kappa_{AC^2} = 0 \ . \tag{65}$$

This analysis should be applied once again, this time for κ_{A^2C} term in (62). Let's start by observing how this term appears in the non-conservation law, once the field $C^{(1)}$ is promoted (gauged) to the dynamical $c^{(1)}$.

$$\mathrm{d} * j_A^{(1)} \supset -\frac{i\kappa_{A^2C}}{8\pi^2} F_A^{(2)} \wedge f_c^{(2)} \tag{66}$$

The current is obviously not conserved, unless $F_A^{(2)}$ is trivial. But, we will show that there is an appropriate source (field) for $f_c^{(2)}$ that will cancel the anomaly and will do so if it undergoes a 2-group shift when $U(1)_A^{(0)}$ transformation is applied. This shows that a $U(1)_A^{(0)} \times U(1)_C^{(0)}$ symmetry, when gauged, births a theory with a 2-group symmetry.
To see exactly how, let's set up a $U(1)_B^{(1)}$ symmetry with a 2-form background gauge field $B^{(2)}$ that is associated with the gauge field strength $f_c^{(2)}$ as the current $J_B^{(2)}$:

$$J_B^{(2)} = \frac{i}{2\pi} * f_c^{(2)} \ , \tag{67}$$

is conserved due to the Bianchi identity that $f_c^{(2)}$ is earlier said to satisfy. Therefore, if we want the current $J_B^{(2)}$ to be defined as above, its classical source should be a background gauge field $B^{(2)}$ with its own $U(1)_B^{(1)}$ background gauge transformation:

$$B^{(2)} \to B^{(2)} + \mathrm{d}\Lambda_B^{(1)} \ , \tag{68}$$

which includes the Bianchi identity for $f_c^{(2)}$. The action should, then, contain:

$$S \supset \int B^{(2)} \wedge *J_B^{(2)} = \frac{i}{2\pi} \int B^{(2)} \wedge f_c^{(2)} \ . \tag{69}$$

The resolution of the anomaly given in (66) is the following: we can impose that $B^{(2)}$ undergoes a 2-group shift under $U(1)_A^{(0)}$ background gauge transformation, and, since $B^{(2)}$ is an appropriate source for $f_c^{(2)}$, if the transformation is of the following form:

$$B^{(2)} \to B^{(2)} + \frac{\hat{\kappa_A}}{2\pi} \Lambda_A^{(0)} F_A^{(2)}, \tag{70}$$

where

$$\hat{\kappa_A} = -\frac{1}{2} \kappa_{A^2C} \ , \tag{71}$$

then the operator-valued shift given in (66) is canceled. Although the transformation rule for θ given with (64) and the transformation rule for $B^{(2)}$ might seem to take a similar form, there is a difference: $B^{(2)}$ transforms only if $F_A^{(2)}$ is non-trivial. This is why, in the first case, the symmetry was explicitly broken and we had to set κ_{AC^2} to be zero, unlike here, where the current $j_A^{(2)}$ is conserved if the field strength $F_A^{(2)} = 0$.

5 Conclusion

Because they lead to conservation laws, continuous global symmetries play an important role in physics. As research of higher-form gauge fields became usual in physics and mathematics, a generalization of symmetry principles to objects of higher dimensions was needed. A q-form global symmetry leads to a $(q+1)$-form conserved current and a conserved charge that has q spatial dimensions. In an attempt to quantize a classical theory with a global symmetry, anomalies can occur. If we, upon quantization, come across a 't Hooft anomaly, the theory cannot be gauged. Having discussed generalized global symmetries through an example of free Maxwell theory - a free $U(1)_c^{(0)}$ gauge theory with two 1-form global symmetries $U(1)_{e,m}^{(1)}$, we have also found the dual formulation of the theory with a matching 't Hooft anomaly.

A 2 -group symmetry is a global symmetry where the mixing of background gauge fields under their respective gauge transformations is involved. The notion of such symmetries was described here using the simplest case: a theory with abelian 2- group symmetry $U(1)_A^{(0)} \times_{\hat{\kappa}_A} U(1)_B^{(1)}$. We have presented how such symmetry arises from an ordinary product symmetry $U(1)_A^{(0)} \times U(1)_C^{(0)}$ by promoting the background gauge field $C^{(1)}$ to a dynamical one. In our attempt to do so, we have analyzed the anomalies that occurred. The generalization of global symmetries is a recent field of research with consequences mainly in string theory and condensed matter physics and offers an organized layout of symmetry principles.

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