2017/2018 (5.) $\lambda_0 \in \sigma(A)$, $Ax = \lambda_0 x$, $x \neq 0$

(a) Dokažimo indukcijom da je $A^k x = \lambda_0^k x \quad \forall k \in \mathbb{N} \cup \{0\}$

BAZA: $k=0 \qquad A^0 = I \qquad Ix = x = \lambda_0^0 \cdot x, \; x \neq 0$

$\Longrightarrow$ x je sv. vektor za $A^0 = I$ pridružen sv. vrijednosti $\lambda_0^0 = 1$.

PRETPOSTAVKA: Pretpostavimo da je $A^k x = \lambda_0^k x$, za neki $k \in \mathbb{N} \cup \{0\}$

KORAK: $A^{k+1} x = A^k(Ax) = A^k(\lambda_0 x) \underset{A^k \in L(V)}{=} \lambda_0 A^k x \underset{\text{induktivna pretpostavka}}{=} \lambda_0(\lambda_0^k x)$

$= \lambda_0^{k+1} x$

$$\left.\begin{array}{l} A^{k+1} x = \lambda_0^{k+1} x \\ x \neq 0 \end{array}\right\} \Longrightarrow$$ x je sv. vektor za $A^{k+1} \in L(V)$ pridružen sv. vrijednosti λ_0^{k+1}!

(b) $p(t) = a_n t^n + a_{n-1} t^{n-1} + \ldots + a_1 t + a_0$

$p(A) = 0 \Rightarrow \quad a_n A^n + a_{n-1} A^{n-1} + \ldots + a_1 A + a_0 I = 0$

$\Rightarrow \quad a_n A^n(y) + a_{n-1} A^{n-1}(y) + \ldots + a_1 A(y) + a_0 y = 0_V \quad \forall y \in V$

$\to x$ je iz a) dijela zadatka

Specijalno, za $y = x$ dobijemo

$$a_n A^n x + a_{n-1} A^{n-1} x + \dots + a_1 A x + a_0 x = 0_V$$

a) dio

$$\Longrightarrow \quad a_n(\lambda_0^n x) + a_{n-1}(\lambda_0^{n-1} x) + \dots + a_1(\lambda_0 x) + a_0 x = 0_V$$

$$\Longrightarrow \quad \left(a_n \lambda_0^n + a_{n-1}\lambda_0^{n-1} + \dots + a_1 \lambda_0 + a_0\right) x = 0_V$$

$$\Longrightarrow \quad \left.\begin{array}{l} p(\lambda_0) x = 0_V \\ x \neq 0 \end{array}\right\} \Longrightarrow \boxed{p(\lambda_0) = 0}$$

2015/16 3. $C : \mathcal{P}_3 \to \mathcal{P}_3$

$$(Cp)(t) = 2p(t) - t p'(t) + 3t p''(t), \quad p \in \mathcal{P}_3$$

Odredimo matrični prikaz operatora C u kanonskoj bazi za $\mathcal{P}_3$ $(e) = \{p_0, p_1, p_2, p_3\}$

$p_0(t) = 1$

$p_1(t) = t$

$p_2(t) = t^2$

$p_3(t) = t^3$

$$(Cp_0)(t) = 2p_0(t) - t p_0'(t) + 3t p_0''(t)$$

$$= 2 - t \cdot 0 + 3t \cdot 0 = 2$$

$$(Cp_1)(t) = 2t - t \cdot 1 + 3t \cdot 0 = t$$

$$(Cp_2)(t) = 2t^2 - t \cdot (2t) + 3t \cdot 2 = 6t$$

$$(Cp_3)(t) = 2t^3 - t \cdot (3t^2) + 3t \cdot (6t) = -t^3 + 18t^2$$

$$\Longrightarrow \quad CP_0 = 2P_0$$

$$CP_1 = P_1$$

$$CP_2 = 6P_1 \qquad \Longrightarrow \quad C(e) = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 6 & 0 \\ 0 & 0 & 0 & 18 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$CP_3 = 18P_2 - P_3$$

$$k_C(\lambda) = \begin{vmatrix} 2-\lambda & 0 & 0 & 0 \\ 0 & 1-\lambda & 6 & 0 \\ 0 & 0 & -\lambda & 18 \\ 0 & 0 & 0 & -1-\lambda \end{vmatrix} = (-1-\lambda) \begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 1-\lambda & 6 \\ 0 & 0 & -\lambda \end{vmatrix}$$

$$= (-1-\lambda)(2-\lambda)(1-\lambda)(-\lambda) = \lambda(\lambda+1)(\lambda-2)(\lambda-1)$$

$$\Longrightarrow \sigma(C) = \{0, 1, -1, 2\}$$

$$a(0) = a(1) = a(-1) = a(2) = 1$$

Kako je $1 \leq g(0) \leq a(0) = 1 \Longrightarrow g(0) = 1$

Analogno zaključujemo da je $g(1) = g(-1) = g(2) = 1$

Kako je $\forall \lambda \in \sigma(C) \quad g(\lambda) = a(\lambda) \Longrightarrow C$ se može dijagonalizirati.

Odredimo svojstvene vektore za sve sv. vrijednosti:

- $Cx = 0$

$$\begin{array}{c}\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 6 & 0 \\ 0 & 0 & 0 & 18 \\ 0 & 0 & 0 & -1 \end{bmatrix}\end{array}\begin{array}{l} \\ \\ \\ /\cdot(18) \; \oplus \end{array} \sim \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}\begin{array}{l} /:2 \\ \\ \\ /\cdot(-1) \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Cx = 0 \iff \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0 \iff x_1 = 0,\ x_2 - 6x_3 = 0,\ x_4 = 0$$

$$V_c(0) = \left\{ \begin{bmatrix} 0 \\ 6t \\ t \\ 0 \end{bmatrix}_{(e)} \middle|\ t \in \mathbb{R} \right\} = \left[\left\{ \begin{bmatrix} 0 \\ 6 \\ 1 \\ 0 \end{bmatrix}_{(e)} \right\} \right] = \left[\{ 6t + t^2 \} \right]$$

- $Cx = x \iff (C - I)x = 0$

$$C - I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & -1 & 18 \\ 0 & 0 & 0 & -2 \end{bmatrix}\begin{array}{l} \\ /:6 \\ \\ /:(-2) \end{array} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 18 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{array}{l} \\ \oplus \\ \\ \\ \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 18 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{array}{l} \\ \\ \\ /\cdot(-18) \; \oplus \end{array} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Cx = x \iff x_1 = 0,\ x_3 = 0,\ x_4 = 0 \implies V_c(1) = \left[\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}_{(e)} \right\} \right]$$

$$= \left[\{ t \} \right]$$

- $Cx = -x \iff (C+I)x = 0$

$$C+I = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 6 & 0 \\ 0 & 0 & 1 & 18 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} /:3 \\ /:2 \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 18 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$Cx = -x \iff x_1 = 0, \; x_2 + 3x_3 = 0, \; x_3 + 18x_4 = 0$

$x_4 = t, \; x_3 = -18t, \; x_2 = -3x_3 = 54t$

$$V_C(-1) = \left[\left\{ \begin{bmatrix} 0 \\ 54 \\ -18 \\ 1 \end{bmatrix}_{(e)} \right\} \right] = \left[\left\{ 54t - 18t^2 + t^3 \right\} \right]$$

- $Cx = 2x \iff (C-2I)x = 0$

$$C-2I = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 6 & 0 \\ 0 & 0 & -2 & 18 \\ 0 & 0 & 0 & -3 \end{bmatrix} \begin{matrix} \\ /\cdot(-1) \\ /:(-2) \\ /:(-3) \end{matrix} \sim \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -6 & 0 \\ 0 & 0 & 1 & -9 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} \\ \\ \\ /\cdot 9 \end{matrix} \oplus$$

$$\sim \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -6 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} \\ \\ /\cdot 6 \\ \\ \end{matrix} \oplus \sim \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$Cx = 2x \iff x_2 = x_3 = x_4 = 0$

$$V_C(2) = \left[\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}_{(e)} \right\} \right] = \left[\{1\} \right]$$

Stavimo $f_0(t) = 6t + t^2$

$f_1(t) = t$

$f_2(t) = 54t - 18t^2 + t^3$

$f_3(t) = 1$

$$C(f) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$\boxed{2015/2016}$ 5. $A \in L(V, W)$, $r(A) = 1 \Rightarrow \dim \operatorname{Im} A = 1$

Neka je $\{w\}$ jedna baza za $\operatorname{Im} A$. Specijalno, $w \neq 0$.

$\forall v \in V \quad Av \in \operatorname{Im} A \Rightarrow Av = \lambda_v w$ za neki $\lambda_v \in \mathbb{F}$.

Neka je $f: V \to \mathbb{F}$ funkcija definirana sa $f(v) = \lambda_v$. Dokažimo da je $f \in V^*$, tj. da je f linearni operator.

$f(v_1 + v_2) = \lambda_{v_1 + v_2}$

$$A(v_1 + v_2) = \lambda_{v_1+v_2} w$$

$$\|$$

$$A(v_1) + A(v_2) = \lambda_{v_1} w + \lambda_{v_2} w = (\lambda_{v_1} + \lambda_{v_2}) w$$

$\Rightarrow \lambda_{v_1+v_2} w = (\lambda_{v_1} + \lambda_{v_2}) w$

Kako je $w \neq 0 \Rightarrow \lambda_{v_1+v_2} = \lambda_{v_1} + \lambda_{v_2}$, tj.

$$f(v_1 + v_2) = f(v_1) + f(v_2) \qquad (1)$$

$f(\alpha v) = \lambda_{\alpha v}$

$$\left.\begin{array}{l} A(\alpha v) = \lambda_{\alpha v} w \\ \quad \| \\ \alpha A(v) = \alpha \cdot (\lambda_v w) = (\alpha \lambda_v) w \end{array}\right\} \Rightarrow \lambda_{\alpha v} w = (\alpha \lambda_v) w$$

$\Rightarrow \lambda_{\alpha v} = \alpha \lambda_v$ (jer je $w \neq 0$)

$\Rightarrow f(\alpha v) = \alpha f(v)$ (2)

Iz (1) & (2) $\Rightarrow f \in V^*$

Dakle $Av = f(v) w \quad \forall v \in V.$

$\boxed{2014/2015}$ 4. b) $A : \mathcal{P}_2 \to \mathcal{P}_2 \quad A(e) = \begin{bmatrix} -1 & 2 & a \\ 1 & 2 & c \\ b & 0 & 1 \end{bmatrix}$

Pretpostavimo da postoji baza (f) za $\mathcal{P}_2$ t.d. $A(f)$ antisimetrična matrica. Tada je $A(f)$ slična matrici $A(e)$

$\Rightarrow tr(A(f)) = tr(A(e))$

$\Rightarrow tr(A(f)) = 2 \quad (1)$

Međutim, $A(f)$ je antisimetrična pa je $A(f) = \begin{bmatrix} 0 & * & * \\ * & 0 & * \\ * & * & 0 \end{bmatrix}$

$\Rightarrow tr(A(f)) = 0 \quad (2)$

(1) & (2) $\Rightarrow 0 = 2 \Rightarrow \Leftarrow$ Ne postoji baza za (f) t.d. $A(f)$ antisimetrična matrica.