$\boxed{2015/2016}$ 4. $P^2 = P$, $\operatorname{Im} P \perp \operatorname{Ker} P$

2. kolokvij

(a) $\operatorname{Im} P \perp \operatorname{Ker} P$

$$\left\{\begin{array}{l} \operatorname{Im} P \oplus \operatorname{Ker} P \leq V \\ \dim(\operatorname{Im} P \oplus \operatorname{Ker} P) = r(P) + d(P) = \dim V \end{array}\right.$$

$$\Rightarrow \operatorname{Im} P \oplus \operatorname{Ker} P = V$$

Za $x, y \in V$ postoje jedinstveni $y_1 \in \operatorname{Im} P$, $y_2 \in \operatorname{Ker} P$ t.d.

$y = y_1 + y_2$

$$\Rightarrow \langle Px, y \rangle = \langle Px, y_1 + y_2 \rangle = \langle Px, y_1 \rangle + \langle \underbrace{Px}_{\in \operatorname{Im} P}, \underbrace{y_2}_{\in \operatorname{Ker} P} \rangle =$$

$$= \langle Px, y_1 \rangle \qquad (1)$$

$$\langle Px, Py \rangle = \langle Px, Py_1 + \underbrace{Py_2}_{0} \rangle = \langle Px, Py_1 \rangle$$

Kako je $y_1 \in \operatorname{Im} P \Rightarrow y_1 = Pz_1$ za neki $z_1 \in V$

$$\Rightarrow Py_1 = P^2 z_1 \underset{P^2 = P}{=} Pz_1 = y_1$$

$$\Rightarrow \langle Px, Py \rangle = \langle Px, Py_1 \rangle = \langle Px, y_1 \rangle \qquad (2)$$

Iz (1) & (2) $\Rightarrow \langle Px, y \rangle = \langle Px, Py \rangle$

(b) Iz a) dijela $\Rightarrow$ $\langle Px, y\rangle = \langle Px, Py\rangle \quad \forall x, y \in V$

$$\langle P^* x, y\rangle = \langle x, Py\rangle = \overline{\langle Py, x\rangle} \underset{(a)}{=} \overline{\langle Py, Px\rangle} = \langle Px, Py\rangle \underset{(a)}{=} \langle Px, y\rangle$$

$\Rightarrow \langle (P^* - P)x, y\rangle = 0 \quad \forall x, y \in V$

Specijalno, za $y = (P^* - P)x \Rightarrow \|(P^* - P)(x)\|^2 = 0 \Rightarrow$

$(P^* - P)(x) = 0 \Rightarrow P^* x = Px \quad \forall x \in V$

$\Rightarrow \boxed{P^* = P}$