Linearna algebra 2

4. zadaća

1. Provjerite da su sljedeći skupovi ortogonalni i nadopunite ih do ortogonalne baze:

   a) $\{(5,2,0,11),(-2,5,1,0)\} \subseteq \mathbb{R}^4$,

   b) $\{(6,-3,-2),(1+i,2i,3)\} \subseteq \mathbb{C}^3$,

   c) $\left\{ \begin{bmatrix} 1 & i \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} -4 & 2 \\ 2+i & 1 \end{bmatrix} \right\} \subseteq M_2(\mathbb{C})$,

   d) $\left\{2t, t^2+1\right\} \subseteq \mathcal{P}_2(\mathbb{R})$, sa skalarnim produktom $\langle p \,|\, q\rangle = \int_{-1}^{1} p(t)q(t)dt$

2. Prikažite vektor v u obliku $v = a + b$, gdje je $a \in M$ i $b \in M^{\perp}$. Odredite udaljenost vektora v od potprostora M.

   a) $v = (1,2,1,4), M = \left\{(x_1,x_2,x_3,x_4) \in \mathbb{R}^4 : x_1 + 2x_2 - 2x_3 - x_4 = 0\right\}$,

   b) $v = (1-i, i, 1+2i), M = \left\{(x_1,x_2,x_3) \in \mathbb{C}^3 : ix_1 + (1-i)x_2 = 2x_2 - ix_3 = 0\right\}$,

   c) $v = \begin{bmatrix} i & 0 \\ 1 & i \end{bmatrix}, M = \left[\left\{ \begin{bmatrix} 1 & 0 \\ 2 & 5 \end{bmatrix}, \begin{bmatrix} 2i & i \\ 0 & 2 \end{bmatrix} \right\}\right] \subseteq M_2(\mathbb{C})$,

   d) $v = t^2 + 1, M = \{p \in \mathcal{P}_2(\mathbb{R}) : p'(-1) = p''(-2) = 0\} \subseteq \mathcal{P}_2(\mathbb{R})$, sa skalarnim produktom $\langle p \,|\, q\rangle = \int_0^1 p(t)q(t)dt$

3. Nađite ortogonalne komplemente sljedećih potprostora i za njih odredite neku ortogonalnu bazu.

   a) $M = \left\{ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{R}) : b + c = 0, a + 2b + 3c + 4d = 0 \right\}$,

   b) $M = \left\{A \in M_2(\mathbb{R}) : A + A^T = \mathrm{tr}(A) \cdot I\right\}$,

   c) $M = \left[\left\{t^2, t^2 - t\right\}\right] \subseteq \mathcal{P}_2(\mathbb{R})$ sa skalarnim produktom $\langle p \mid q\rangle = \int_{-1}^{1} p(t)q(t)dt$,

   d) $M = [\{p \in \mathcal{P}_2(\mathbb{R}) : p'(t) = 6t - 1\}]$ sa skalarnim produktom

   $$\langle p \mid q\rangle = \int_0^1 p(t)q(t)dt,$$

   e) $M = [\{(1,0,1,-2),(-1,1,-1,1),(-1,2,-1,0)\}] \subseteq \mathbb{R}^4$.

4. Prikažite vektor $v = \sin x$ u obliku $v = a + b$, gdje je $a \in M$ i $b \in M^{\perp}$, za $M = [\{1, \cos x\}] \subseteq C([0,\pi];\mathbb{R})$

5. Odredite koja su od sljedećih preslikavanja linearni operatori:

a) $A : \mathbb{R}^3 \to \mathbb{R}^3, A(x_1, x_2, x_3) = (x_2 + x_3, x_1 - x_2 + 2x_3, -2x_1 + x_3)$,

b) $C : \mathbb{R}^3 \to \mathbb{R}^3, C(x_1, x_2, x_3) = (x_2 + x_3, x_1 - x_2 + 2x_3, -2x_1 + x_3x_2)$,

c) $D : \mathbb{C}^2 \to \mathbb{C}, D(x, y) = \langle (x, y) \mid (2 + i, 1) \rangle$, gdje je $\langle \cdot \mid \cdot \rangle$ standardni skalarni produkt na $\mathbb{C}^2$

d) $E : \mathbb{C}^2 \to \mathbb{C}, E(x, y) = \langle (2 + i, 1) \mid (x, y) \rangle$, gdje je $\langle \cdot \mid \cdot \rangle$ standardni skalarni produkt na $\mathbb{C}^2$

e) $F : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}^3, F(p) = (p(0), p'(1), p''(2))$,

f) $G : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}^3, G(p) = \big(p^2(0), p'(1), p''(2)\big)$,

g) $I : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}, I(p) = \int_0^1 p^2(t)dt$

h) $J : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}^2, J(p) = \left(\int_{-1}^1 p(t)dt, t\right)$,

i) $K : M_2(\mathbb{C}) \to \mathbb{C}, K(X) = \operatorname{tr} X$,

j) $M : M_2(\mathbb{C}) \to M_2(\mathbb{C}), M(X) = \langle I \mid X \rangle I$, gdje je I jedinična matrica u $M_2(\mathbb{C})$.

6. Neka je $A : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ linearni operator takav da je

$$A\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix},\ A\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right) = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}.$$

Odredite $A\left(\begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}\right)$ i $A\left(\begin{bmatrix} 5 & 5 \\ 2 & 5 \end{bmatrix}\right)$. Za koje matrice $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{R})$ možemo iz zadanih podataka odrediti $A\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$?

7. Postoji li linearni operator $A : \mathbb{R}^3 \to \mathbb{R}^2$ takav da vrijedi:

(a) $A(1, 2, 3) = (1, 1)$ i $A(2, 4, 6) = (2, 3)$?

(b) $A(1, 0, 0) = (1, 1)$ i $A(2, 0, 0) = (2, 2)$?

(c) $A(1, 0, 1) = (2, 1)$, $A(1, 1, 0) = (1, 2)$, $A(0, 1, -1) = (0, 1)$.

8. Neka je $A : \mathbb{R}^3 \to \mathbb{R}^2$ linearni operator takav da je

$$A(1, 1, 1) = (2, 3),\ A(1, 0, 1) = (1, 1),\ A(0, 1, 1) = (0, 0).$$

Možemo li iz zadanih podataka potpuno odrediti linearni operator A, tj. odrediti $A(x_1, x_2, x_3)$ za proizvoljni vektor $(x_1, x_2, x_3) \in \mathbb{R}^3$?

9. Neka je $\{e_1, e_2\}$ kanonska baza za $\mathbb{R}^2$, $f_1 = (1, 1), f_2 = (1, 2)$. Neka su $A, B : \mathbb{R}^2 \to \mathbb{R}^2$ linearni operatori takvi da je

$$Ae_1 = (1, -1),\ Ae_2 = (1, 0)$$
$$Bf_1 = (2, -1),\ Bf_2 = (3, -1).$$

Obrazložite zašto su ovim podacima linearni operatori A i B jedinstveno određeni. Odredite Af_1, Af_2. Što možete zaključiti o operatorima A i B?