Krešimir Kumerički:

Bilješke s predavanja

Fizika elementarnih čestica I

ak. godina 2016/2017.

Fizika (elementarnih) čestica = fizika visokih energija (HEP)

↓

Demokrit … atomi … nedjeljivost

Mendeljejev:

H		….		He
Li	Be		F	Ne
Na	Mg		Cl	Ar

→ 98% vidljivog svemira (H & He)

↓

ponavljanje istih svojstava (samo masa raste) → objašnjeno građom atoma. (valentne ljuske iste u atomima iste grupe)

Standardni model (SM):

kvarkovi		leptoni	
u	d	ν_e	e
c	s	ν_μ	μ
t	b	ν_τ	τ

(spin = $\hbar/2$) → FERMIONI

→ ≈ 100 % vidljivog svemira (prvi red)

u u d — proton

u d d — neutron

↓

3x ponavljanje istih svojstava (samo masa raste) → NIJE još objašnjeno

temeljne sile:

1. elektromagnetska
2. slaba
3. jaka
4. gravitacijska

prijenosnici:

foton γ

W, Z

gluon G

graviton ?

spin = $\hbar$ → BOZONI

$\boxed{\text{Higgsov bozon } H}$ (spin = 0)

Teorija koja opisuje ponašanje i međudjelovanje čestica:

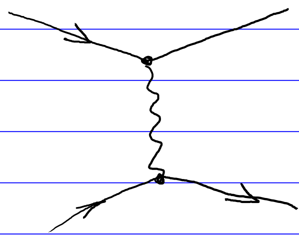
kvantna mehanika + specijalna teorija relativnosti = kvantna teorija polja (QFT)
(QM) (STR)

$\Downarrow$

(debela knjiga / izborni kolegiji)

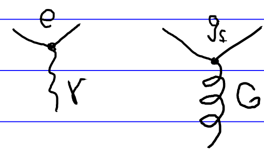
$\Downarrow$

QM amplituda = Feynmanovi dijagrami



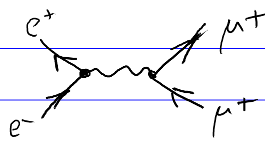
Svakoj čestici odgovara linija

Svakoj sili (međudjelovanju) odgovara vrh

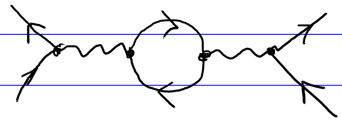


Amplituda procesa: svi mogući dijagrami (kombinatorika + račun smetnje)

$\mathcal{M}(e^+e^- \to \mu^+\mu^-) =$



$+$



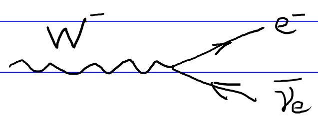
$+ \ldots$

$\propto e^2$

$\propto e^4$ pa možemo zanemariti za mali e

Ovo je primjer raspršenja dviju čestica. Jedan od dva glavna procesa u FEČ. Drugi je raspad:

npr.



Detekcija čestica

Tipično srednje vrijeme života : $\tau = \begin{cases} 2.2 \times 10^{-6}\, s & (\mu^{\pm}) \\ 2.6 \times 10^{-8}\, s & (\pi^{\pm}) \\ 8.4 \times 10^{-17}\, s & (\pi^{0}) \end{cases}$
(u sustavu čestice)

Tipična brzina : $v = 0.998\,c$

Srednji prevaljeni put : $L = v \gamma \tau$

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1 - 0.998^2}} = 15.8$$

$$\Rightarrow \quad L = \begin{cases} 10 \text{ km} \\ 100 \text{ m} \\ 1\ \mu\text{m} \end{cases}$$

Dakle relativno mali broj čestica se može izravno detektirati :

elektron, proton, foton, neutrino $\rightarrow$ stabilne

neutron, mion, nabijeni pion, kaon $\rightarrow$ $\tau \gtrsim 10^{-10}\, s$

Ostale čestice opažamo indirektno, putem čestica u koje se raspadaju.

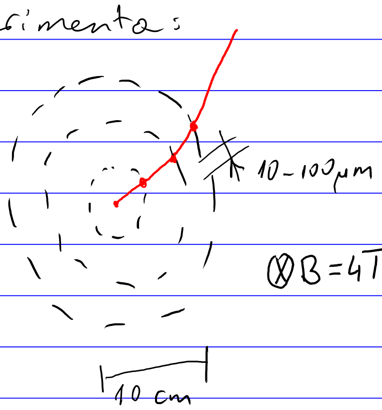
Detektori tragova (trackers)

Nabijene čestice velike energije ioniziraju tvar kroz koju prolaze.

- prezasićena para (Wilson cloud chamber)
- emulzija
- pregrijana tekućina (bubble chamber)
- plin
- poluvodič

(4 Nobelove nagrade)

Npr. unutarnji poluvodički detektor tragova (pixel detektor) CMS eksperimenta:



Iz radijusa zakrivljenosti R odredimo impuls čestice:

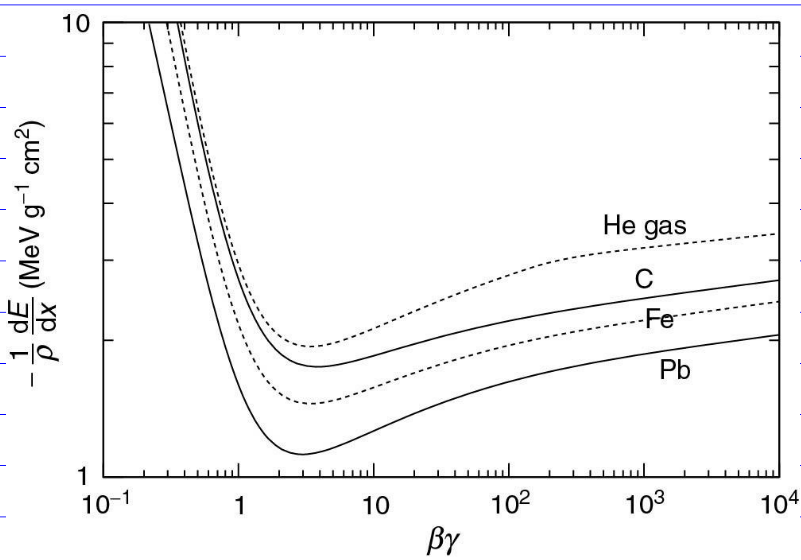
$$F_{Lorentz} = F_{centripetal.}$$

$$q|\vec{v} \times \vec{B}| = q v_\perp B = \frac{m v_\perp^2}{R} \qquad (\perp \equiv \text{okomito na } \vec{B})$$

$$m v_\perp = \boxed{p_\perp = qBR} \qquad (\text{SI sustav})$$

vrijedi i u relativističkom slučaju

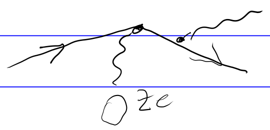
Ionizacija jednostruko nabijene čestice:



← Bethe-Bloch formula

(vidi Praktikum iz moderne fizike)

Nakon kritične energije ($E_c = (800/Z)$ MeV za e^-, $E_c \sim 100$ GeV za μ^-) zakočno zračenje (bremsstrahlung) je značajnije od ionizacije.

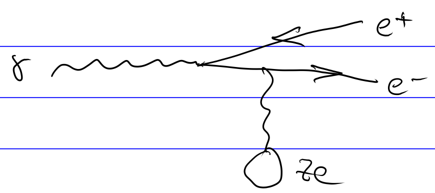


Interakcija fotona s materijom (cf. Praktikum)

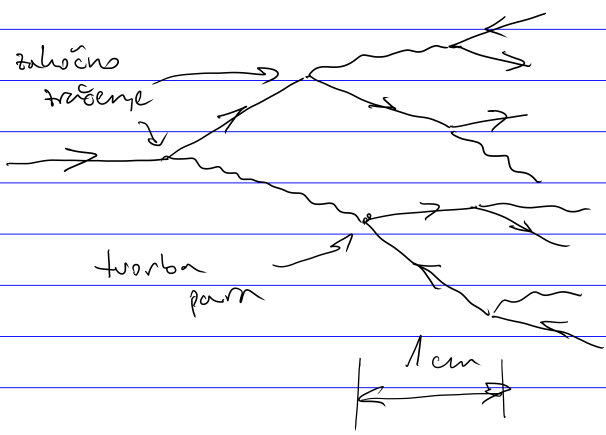
Energija

- Rayleighovo raspršenje
- fotoefekt
- komptonsko raspršenje
- tvorba para
- fotonuklearne interakcije

dominantna za $E > \text{GeV}$



u kombinaciji s zakočnim zračenjem tvorba para dovodi do elektromagnetskih pljuskova:



detektiramo se scintilacijski fotoni

$\propto E$.

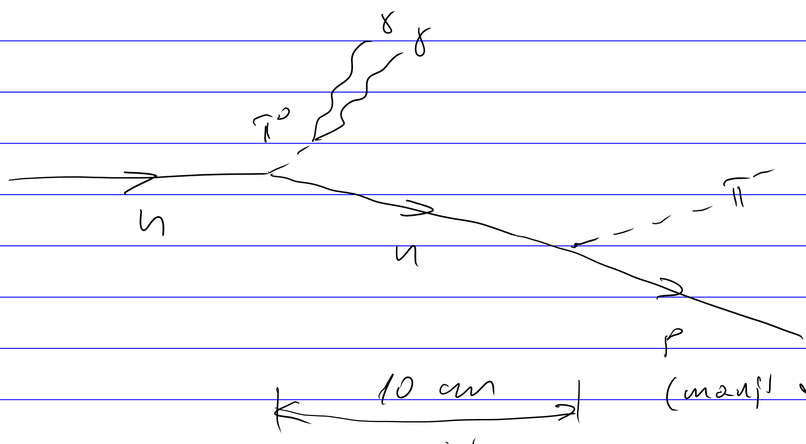
↓

druga velika grupa detektora:

Kalorimetri

$e^-, \gamma \to$ em pljuskovi $\to$ ECAL

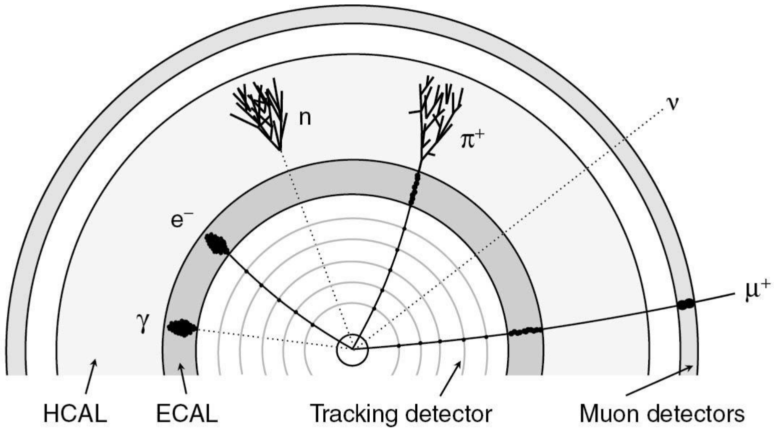
$n, \pi^\pm \to$ hadronski pljuskovi $\to$ HCAL



(manji doseg nuklearnih sila)

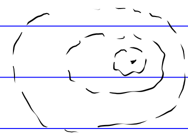
$\Rightarrow$ HCAL > ECAL

Tipične građa detektora opće namjene:



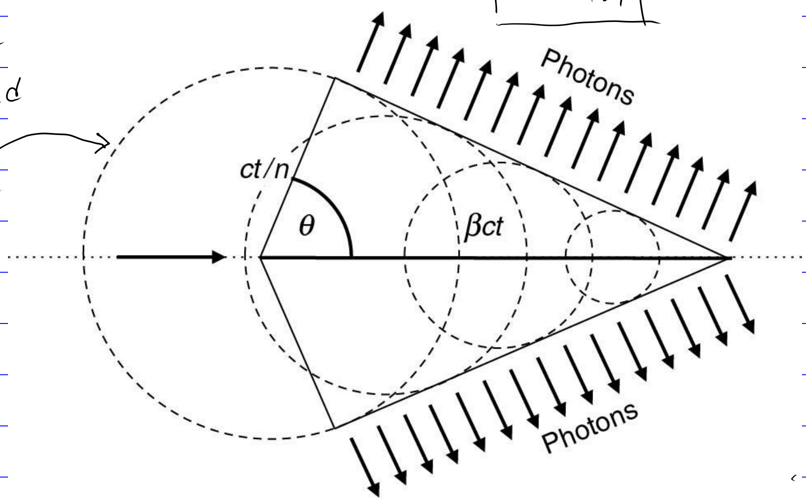
Čerenkovljevi detektori

$v < \frac{c}{n}$

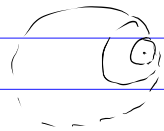


$\boxed{v > c/n}$

valna fronta fotona uslijed depolarizacije sredstva



$v = c/n$



$$\cos\vartheta = \frac{ct/n}{vt} = \frac{1}{n\beta}$$

n – indeks loma

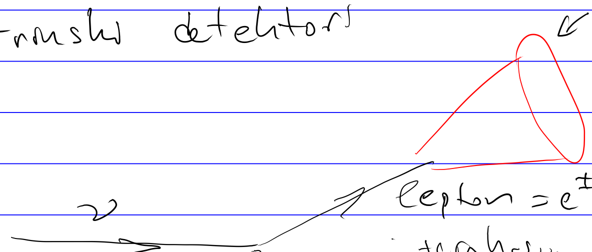
$\beta = v/c$ – brzina čestice

– neutrinski detektori

prsten na detektorima fotona (RICH)

↓

možemo identificirati česticu



ν

lepton $= e^{\pm}, \mu^{\pm}$

slaba interakcija sa sredstvom

– za akceleratore, luminozitet, udarni presjek → VJEŽBE

Prirodni sustav jedinica

Fizikalne veličine relevantne za FEČ i njihove mjerne jedinice u SI sustavu:

masa [kg], naboj [C]

vrijeme raspada [s], udarni presjek [m^2]

Jedinice za energiju, impuls itd su izvedene od ovih.

s – brojanje titraja Cs

m – $c \stackrel{def}{=} 3.00X \times 10^8$ m/s fiksirano → definira m

kg – SI: etalon

novi SI: $h \stackrel{def}{=} 6.626X \cdot 10^{-34}$ kg m^2/s fiksirano → definira kg

Želimo opisati jako male i jako velike brojeve:

udarni presjek: 1 b (barn) $= 10^{-28}$ m^2

Nismo si puno pomogli (fb i ab se uredno koriste).

Ono što pomaže je prirodni sustav jedinica:

c ionako definira metar. Idemo korak dalje:

– mjerimo brzinu u jedinicama c umjesto m/s
– mjerimo moment impulsa u $\hbar$. npr. spin elektrona $= 1/2$

Da bi se riješili kompletne makro-trojke kg, m, s treba nam treća konstanta ili pogodna jedinica

Newtonova G_N bi matematički gledano poslužila ali → nepreciznno mjerena → nije numerički pogodna za FEČ (nema veze s FEČ)

Biramo:

$$\boxed{1\ \mathrm{GeV}} = 1.6 \times 10^{-10}\ \mathrm{J}\ \text{tj.}\ \mathrm{kg}\ \mathrm{m}^2/\mathrm{s}^2$$

Energiju mjerimo u GeV.

Masa? $[\mathrm{GeV}] = \mathrm{kg}\,(\mathrm{m}/\mathrm{s})^2 \Rightarrow [\text{masa}] = \dfrac{\mathrm{GeV}}{c^2}$

npr. $m_{proton} = 0.938\ \mathrm{GeV}/c^2$

Vrijeme? $[\hbar] = \mathrm{J\,s}$ tj. $\mathrm{GeV\,s} \Rightarrow [\text{vrijeme}] = \hbar/\mathrm{GeV}$

Udaljenost se mjeri u $\hbar c/\mathrm{GeV}$.

Daljnje pojednostavljenje postižemo izborom

$$\boxed{\hbar = c = 1}$$

pa se sve mjeri u GeV! Tek na kraju, dimenzionalnom analizom rekonstruiramo $\hbar$ i c u rezultatu i po potrebi pretvorimo nazad u SI sustav jedinica.

Primjer: radijus protona

$r = 4.1 \text{ GeV}^{-1}$

$r = 4.1 \text{ GeV}^{-1} \hbar^{\alpha} c^{\beta}$

$$[r] = \left(\frac{kg\, m^2}{s^2}\right)^{-1} \underbrace{\left(\frac{kg\, m^2}{s^2} \cdot s\right)}_{Js}^{\alpha} \left(\frac{m}{s}\right)^{\beta} = m \, (\cdot kg^0 \cdot s^0)$$

$-2 + 2\alpha + \beta = 1$ ✓

$-1 + \alpha = 0 \quad \rightarrow \alpha = 1$

$2 - \alpha - \beta = 0 \quad \rightarrow \beta = 1$

$$r = 4.1 \text{ GeV}^{-1} \hbar c = 4.1 \frac{1.055 \times 10^{-34} \, \cancel{J}s \cdot 3 \times 10^8 \, m/\cancel{s}}{1.602 \times 10^{-10} \, \cancel{J}} = 0.81 \underbrace{\times 10^{-15} m}_{fm}$$

Za brži račun pogodni su konverzijski faktori poput

$$\boxed{\hbar c = 0.197 \text{ GeV fm}}$$ (= "1" u sustavu $\hbar = c = 1$ pa to smijemo ubaciti gdje želimo)

$r = 4.1 \, \cancel{\text{GeV}}^{-1} \cdot 0.197 \, \cancel{\text{GeV}} \text{ fm} =$

$= 0.81 \text{ fm}$

Pa se $\hbar$ i c zapravo nikad ne pojavljuju u računima u FEČ!

I na kraju biramo još

$$\boxed{\varepsilon_0 = \mu_0 = 1}$$ (Heaviside-Lorentzov sustav jedinica)

- $[\varepsilon_0]$ uključuje naboj koji još nismo takli pa imamo tu slobodu

- kad odaberemo $\varepsilon_0 = 1 \Rightarrow \mu_0 = 1/c^2\varepsilon_0 \rightarrow 1$

Primjer: konstanta fine strukture

$$\alpha = \begin{cases} \dfrac{e^2}{4\pi \varepsilon_0 \hbar c} & (SI) = \dfrac{1}{137} \\ \dfrac{e^2}{4\pi} & (\text{prirodni sustav}) = \dfrac{1}{137} \end{cases}$$

Relativistička kinematika

kontravarijantne komponente

četvero-vektor impulsa $p^\mu = (E, p_x, p_y, p_z) \equiv (E, \vec{p})$

energija, impuls

$p_\mu = g_{\mu\nu} p^\nu = (E, -\vec{p})$ gdje koristimo „west coast" metriku $g = \mathrm{diag}(1, -1, -1, -1)$

(„east coast" metrika $g = \mathrm{diag}(-1, 1, 1, 1)$ je popularnija u literaturi iz gravitacije)

$a \cdot b = a_\mu b^\mu$ je relativistička invarijanta, jednako u svim (inercijalnim) sustavima

Primjer: čestica mase m, impuls p^μ

$p^2 = p \cdot p = p_\mu p^\mu = E^2 - \vec{p}^{\,2}$

za izvrijedniti biramo sustav središta impulsa : $\boxed{\vec{p}^{\,CM}_{ukupno} = 0}$

(CM ≡ „center of momentum" ≠ centar mase)

$E_{CM} = (\text{masa mirovanja}) = m \qquad (E = mc^2, \; c=1)$

Thomson: $E_{CM} \equiv E^*$

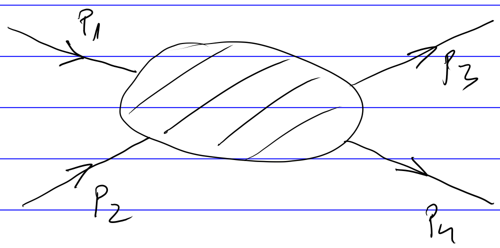
$\rightarrow \quad p^2 = m^2 \qquad$ (vrijedi u svim sustavima)

Za višečestične sustave, npr. dvije čestice s impulsima p_1, p_2, također koristimo izraz „invarijantna masa":

$$m_{12}^2 = (p_1 + p_2)^2$$

Mandelstamove varijable

Za općenito raspršenje dvije čestice u dvije čestice

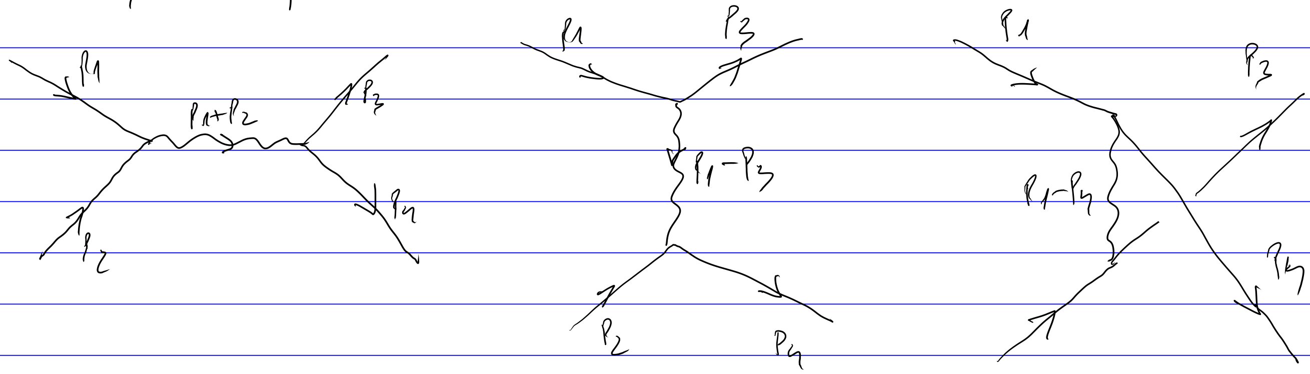


pogodno je koristiti relativističke invarijante

$$\boxed{\begin{aligned} s &\equiv (p_1+p_2)^2 \\ t &\equiv (p_1-p_3)^2 \\ u &\equiv (p_1-p_4)^2 \end{aligned}}$$

(Mandelstamove varijable)

Ovakva definicija motivirana je mogućnostima međudjelovanja izmjenom jedne čestice:



"s-kanal" "t-kanal" "u-kanal"

Vrijedi $s+t+u=\sum_{i=1}^{4} m_i^2$ pa su samo dvije od tri Mandelstamove varijable nezavisne.

Amplitude i udarni presjeci se često izražavaju preko s,t,u invarijanti i tek na kraju se pretvaraju u kuteve i energije (koje se mjere u eksperimentu).

$$s=(p_1+p_2)^2=(E_1+E_2,\vec{p}_1+\vec{p}_2)^2=(E_1^{CM}+E_2^{CM},0)^2=(E_1^{CM}+E_2^{CM})^2$$

$\Rightarrow \sqrt{s}=E_1^{CM}+E_2^{CM}$ – ukupna energija u centru impulsa (na raspolaganju za proizvodnju novih čestica)

LHC: $E_1^{CM}=E_2^{CM}=6.5\ \text{TeV}$, $\sqrt{s}=13\ \text{TeV}$

Fermijevo zlatno pravilo

Nerelativistička (NR) kvantna mehanika (QM): vremenski ovisan račun smetnje u prvom približenju:

Brzina prijelaza $\underset{\text{"initial"}}{|i\rangle} \to \underset{\text{"final"}}{|f\rangle}$ (vjerojatnost prijelaza u jedinici vremena)

dano je s

$$\Gamma_{fi} = \frac{2\pi}{\hbar} \underbrace{|\langle f | H_{int} | i \rangle|}_{\equiv T_{fi}}{}^2 \rho(E_i)$$

(Fermijevo zlatno pravilo)

T_{fi} je element matrice prijelaza. Bit će dan Feynmanovim pravilima.

$\rho(E_i)$ je gustoća konačnih (kvantnih) stanja

$$\rho(E_i) \equiv \left|\frac{dn}{dE}\right|_{E=E_i}$$

dn je broj dostupnih stanja s energijom od E do $E+dE$

$$\left|\frac{dn}{dE}\right|_{E=E_i} = \int \frac{dn}{dE}\, \delta(E_i - E)\, dE$$

pa je

$$\Gamma_{fi} = 2\pi \int |T_{fi}|^2 \, \delta(E_i - E)\, dn$$

(alternativni zapis Fermijevog zlatnog pravila)

Trebamo naučiti brojati $\underbrace{\text{dostupna stanja sustava}}_{\equiv \text{ dostupni fazni prostor}}$ relativističkih čestica

(prostor čije koordinate potpuno specificiraju stanje sustava)

Fazni prostor sustava čestica

Specifikacija elementarne čestice: $\underbrace{\text{masa}(m), \text{spin}(J)}_{\text{prostorno-vremenske „labele" (JEDINE!)}}$, $\underbrace{\text{naboj (električni i dr.)}}_{\text{interne labele}}$

(Bit će važno za T_{fi}, ali nebitno za $S(E)$.)

Čestice daleko od područja interakcije (npr. netom prije detekcije) možemo smatrati slobodnima.

Specifikacija stanja slobodne elementarne čestice mase m i spina J:

$$\underbrace{\text{tro-impuls } \vec{p} \in R^3 \text{ i orijentacija spina (polarizacija) } J_3 \in \underbrace{\{-J, -J+1, \dots J\}}_{2J+1 \text{ stanja}}}_{\text{Utjecajem polarizacije na } T_{fi} \text{ pozabavit ćemo se kasnije.}}$$

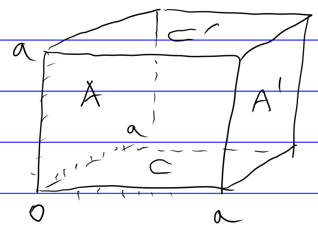
$$\psi(x) = A e^{-ip\cdot x} = A e^{-i(Et - \vec{p}\cdot\vec{x})}$$

Normalizacija uobičajena u NR QM je jedna čestica u volumenu V.
V – proizvoljan (dovoljno velik), mjerene veličine ne smiju ovisiti o njemu

$$\int_V \underbrace{\psi^*(x)\,\psi(x)}_{\rho(x) \text{ - gustoća broja čestica}} d^3x = 1$$

$$\rho(x) = A^2 \quad \Rightarrow \quad A^2 V = 1 \quad \Rightarrow \quad A = 1/\sqrt{V}$$

Rubni uvjet: na rubu volumena V je $\psi(\vec{x}) = 0$.
Ekvivalentno, a lakše za računati: periodički rubni uvjet



je hipertorus $A \equiv A'$, $B = B'$, $C = C'$

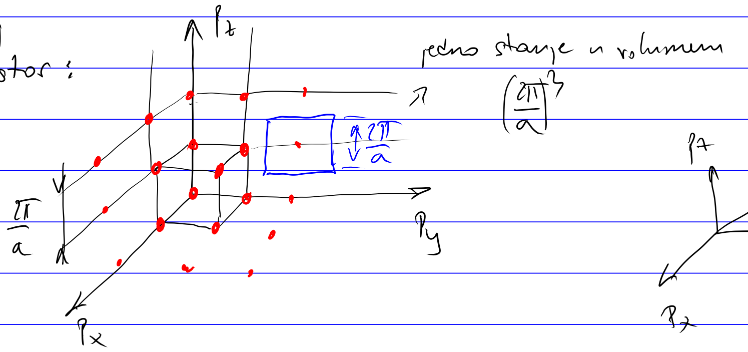
$$\Rightarrow \psi(x,y,z) = \psi(x+a,y,z) = \psi(x,y+a,z) = \dots$$

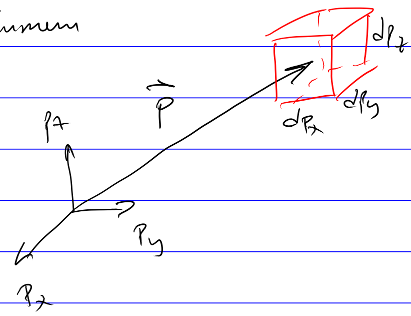
$$\rightarrow e^{ip_x x} = e^{ip_x(x+a)} \Rightarrow e^{ip_x a} = 1 \Rightarrow p_x a = n_x 2\pi$$

$V = a^3$

$$(p_x, p_y, p_z) = (n_x, n_y, n_z)\,\frac{2\pi}{a}$$

Fazni prostor:





Broj stanja dn u diferencijalnom volumenu $dp_x\, dp_y\, dp_z = d^3\vec p$

je gustoća × volumen $= \frac{1}{(2\pi)^3/V} \cdot d^3\vec p = \frac{V\, d^3\vec p}{(2\pi)^3}$

T_{fi} ne smije ovisiti o izboru V. Faktori V za svaku česticu će se pokratiti s odgovarajućim faktorima normalizacije $\psi(x)$ u $|T_{fi}|^2 \sim |\langle f|H|i\rangle|^2 \sim \left|\int \left(\frac{1}{\sqrt{V}} e^{-ip\cdot x}\right) \cdots \right|^2$

(Zapravo, dokaz ovog kraćenja je nešto kompliciraniji, vidi ozbiljnije knjige.)

Stoga biramo $V=1$. No, prije toga još jedna finesa:

Normalizacija $\int_V d^3r\, \psi^*(x)\psi(x) = 1$ tj. jedna čestica po (jediničnom) volumenu V nije Lorentz-invarijantna!

Potisak za $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ komprimira volumen na V/γ

Lorentz invarijantna normalizacija može se postići izborom valne funkcije $\psi'(x)$ normalizirane kao

$$\int_V \psi'(x)\,\psi(x)\, d^3x = 2E$$ tj. 2E čestica po jediničnom volumenu

Potisak radi $E \to \gamma E$ pa je smanjenje volumena kompenzirano povećanjem energije.

Konkretiziramo formule za raspad $a \to 1 + 2$

$$\Gamma_{fi} = 2\pi \int |T_{fi}|^2 \delta(E_i - E) \frac{d^3\vec{p}_1}{(2\pi)^3} \qquad E_i = E_a$$

– ne smijemo dodati $d^3\vec{p}_2$ jer zbog zakona očuvanja $p_a = p_1 + p_2$

specifikacija $\vec{p}_1$ potpuno određuje konačno stanje. ($\vec{p}_2$ je fiksiran kao $\vec{p}_2 = \vec{p}_a - \vec{p}_1$)

Radi lakšeg poopćenja ubacimo u integral $\delta^3(\vec{p}_a - \vec{p}_1 - \vec{p}_2)\, d^3\vec{p}_2 \to 1$

$$\Gamma_{fi} = (2\pi)^4 \int |T_{fi}|^2 \delta(E_a - E_1 - E_2)\, \delta^3(\vec{p}_a - \vec{p}_1 - \vec{p}_2) \frac{d^3\vec{p}_1}{(2\pi)^3} \frac{d^3\vec{p}_2}{(2\pi)^3}$$

Usporedba normalizacija

$\psi = \frac{1}{\sqrt{2E}} \psi'$ sugerira:

$$|T_{fi}|^2 = |\langle \psi_1 \psi_2 | H_{int} | \psi_a \rangle|^2 = \frac{1}{(2E_a)(2E_1)(2E_2)} |\underbrace{\langle \psi_1' \psi_2' | H_{int} | \psi_a' \rangle}_{\equiv \mathcal{M}_{fi}}|^2$$

$$\boxed{\Gamma_{fi} = \frac{1}{2E_a} \int |\mathcal{M}_{fi}|^2 \underbrace{(2\pi)^4 \delta^4(p_a - p_1 - p_2) \frac{d^3\vec{p}_1}{(2\pi)^3 2E} \frac{d^3\vec{p}_2}{(2\pi)^3 2E}}_{dLips}}$$

Fermijevo zlatno pravilo u Lorentz-invarijantnom obliku

dLips ("Lorentz-invariant phase space")

(pokazano na vježbama da je $\frac{d^3\vec{p}}{E}$ invarijanta)

Lorentz-invarijantni matrični element a.k.a Feynmanova amplituda

Raspadi čestica – općenito

Nestabilna čestica se tipično može raspasti na više različitih konačnih stanja, tzv. kanala

Npr.

$$H \to e^+e^- \quad \text{s brzinom prijelaza} \quad \Gamma_1$$
$$\to \gamma\gamma \quad \text{-ll-} \quad \Gamma_2$$
$$\to b\bar{b} \quad \text{-ll-} \quad \Gamma_3$$
$$(\dots) \quad (\dots) \quad (\dots)$$

parcijalne brzine raspada
a.k.a. -ll- širine -ll-

Za vrijeme δt vjerojatnost raspada u kanal s parcijalnom širinom Γ_k je $(\delta t \Gamma_k)$; ako imamo N čestica nakon δt prosječno $(N\Gamma_k \delta t)$ će ih se raspasti u taj kanal

Ukupna promjena broja čestica nakon δt:

$$\delta N = -N\Gamma_1 \delta t - N\Gamma_2 \delta t - \dots = -N\,\delta t \sum_k \Gamma_k \equiv -N\Gamma\,\delta t$$

$$\Gamma = \sum_k \Gamma_k \quad \text{– ukupna širina raspada}$$

$$\frac{\delta N}{N} = -\Gamma\,\delta t \quad \Big/ \int_0^t \quad \to \quad \ln N \Big|_0^t = \ln N(t) - \ln N(0) = \ln\frac{N(t)}{N(0)} = -\Gamma t$$

$$\boxed{N(t) = N(0)\, e^{-\Gamma t}} = N(0)\, e^{-t/\tau}$$

$$\tau = \frac{1}{\Gamma} \quad \text{– prosječno vrijeme života}$$

~~Loš argument: Nakon vremena τ vjerojatnost raspada je $\Gamma\tau = 1$ $\Rightarrow$ čestica se nakon τ raspadne s vjerojatnošću $=1$~~

Ispravno: Vjerojatnost da čestica doživi t je $\propto e^{-\Gamma t}$. očekivano vrijeme života je stoga

$$\hat{\tau} = \frac{\int_0^\infty t\, e^{-\Gamma t}\, dt}{\int_0^\infty e^{-\Gamma t}\, dt} = \frac{-\frac{e^{-\Gamma t}(\Gamma t + 1)}{\Gamma^2}\Big|_0^\infty}{-e^{-\Gamma t}/\Gamma \Big|_0^\infty} = \frac{1/\Gamma^2}{1/\Gamma} = \frac{1}{\Gamma}$$

$$BR(k) = \frac{\Gamma_k}{\Gamma} \quad \text{- omjer grananja}$$

Npr. $BR(Z \to e^+ e^-) = 3.36\,\%$

$BR(Z \to \text{neutrini}) = 20\,\%$

$(\ldots)$

$\sum_k BR(k) = 1$ tj. $100\,\%$

Raspad $a \to 1 + 2$ (nastavak)

Širina raspada

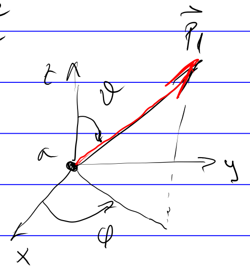
$$\Gamma = \Gamma_{fi} = \frac{1}{2E_a} \int |\mathcal{M}|^2 (2\pi)^4 \delta^{(4)}(p_a - p_1 - p_2) \frac{d^3 p_1}{(2\pi)^3 2E_1} \frac{d^3 p_2}{(2\pi)^3 2E_2}$$

U sustavu mirovanja čestice a: $E_a = m_a$, $\vec{p}_a = 0$

$$\delta^{(4)}(p_a - p_1 - p_2) = \delta(m_a - E_1 - E_2)\, \underbrace{\delta^{(3)}(\vec{p}_1 + \vec{p}_2)}_{\text{Omogućuje trivijalni integral po } d^3\vec{p}_2}$$

$$\Gamma = \frac{1}{(2\pi)^2\, 2m_a} \int |\mathcal{M}|^2 \delta(m_a - E_1 - E_2) \frac{d^3\vec{p}_1}{2E_1 2E_2}$$

gdje je sad svugdje $\vec{p}_2 = -\vec{p}_1$ i $E_2 = \sqrt{\vec{p}_1^{\,2} + m_2^2}$



$$d^3\vec{p}_1 = |\vec{p}_1|^2 d|\vec{p}_1|\, d\Omega \quad , \quad d\Omega = d(\cos\vartheta)\, d\varphi$$

$|\vec{p}_1| \equiv x$

$$\Gamma = \frac{1}{8\pi^2 m_a} \int |\mathcal{M}|^2 \frac{x^2}{4E_1E_2}\, \underbrace{\delta\big(\underbrace{m_a - \sqrt{x^2+m_1^2} - \sqrt{x^2+m_2^2}}_{f(x)}\big)}_{\left|\frac{\partial f}{\partial x}\right|^{-1}_{x=x_0} \delta(x - x_0)}\, dx\, d\Omega$$

x – integracijska varijabla

x_0 – nul-točka od $f(x)$, $f(x_0) = 0$

$x_0 = |\vec{p}_1^{\,(0)}|$ – stvarni impuls izlazne čestice

$$\left|\frac{\partial f}{\partial x}\right| = \frac{x}{\sqrt{x^2+m_1^2}} + \frac{x}{\sqrt{x^2+m_2^2}} = \frac{x}{E_1} + \frac{x}{E_2} = \frac{x(E_1+E_2)}{E_1E_2} =$$

$$\Gamma = \frac{1}{8\pi^2 m_a}\int |\mathcal{M}|^2 \, \frac{x^{\cancel{2}}}{4\cancel{E_1E_2}} \, \frac{\cancel{E_1E_2}}{\cancel{x}(E_1+E_2)} \, \delta(x-x_0)\, d\Omega$$

$$E_1 + E_2 = m_a$$

$$\boxed{\Gamma(a\to 1+2) = \frac{|\vec{p}_1|}{32\pi^2 m_a^2}\int |\mathcal{M}|^2 \, d\Omega}$$

Gdje je $|\vec{p}_1|$ dano zakonima očuvanja tj. rješavanjem $f(x)=0$:

(vježbe:)

$$|\vec{p}_1| = \frac{1}{2m_a}\sqrt{\left[m_a^2-(m_1+m_2)^2\right]\left[m_a^2-(m_1-m_2)^2\right]}$$

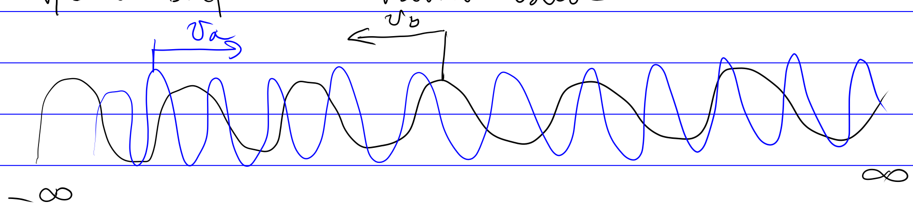
Raspršenja, udarni presjek

Pored raspada $a \to 1+2+\dots$, raspršenje $a+b \to 1+2+\dots$ je drugi važni proces fizike čestice.

Fermijevo zlatno pravilo je u načelu o.u. uz kreiranje još jedne čestice u $|i\rangle$. Npr. za $a+b \to 1+2$:

$$\Gamma_{fi} = \frac{1}{(2E_a)(2E_b)} \int |\mathcal{M}_{fi}|^2 (2\pi)^4 \delta^{(4)}(p_a+p_b-p_1-p_2) \frac{d^3p_1}{(2\pi)^3 2E_1} \frac{d^3p_2}{(2\pi)^3 2E_2}$$

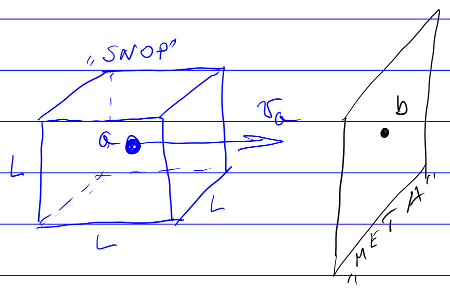
Ali to je brzina reakcije (vjerojatnost raspršenja u sekundi) za dva vječno sveprisutna ravna vala



To nije podesno za opis realnih eksperimenata, gdje čestice interagiraju kratko vrijeme i zanima nas hoće li se raspršiti ili ne.
U tu svrhu definiramo omjer Γ_{fi} s gustoćom toka čestica ϕ:

udarni presjek $$\sigma = \frac{\Gamma_{fi}}{\phi} = \frac{\text{vjerojatnost reakcije u jed. vremenu}}{\text{tok čestica po jed. površine u jed. vremenu}}$$

Dimenzija:
$$[\sigma] = \frac{T^{-1}}{\# L^{-2} T^{-1}} = L^2 \quad - \text{površina}$$



U nekom vremenu T broj čestica koje prolaze metu je $\frac{v_a T}{L}$
pa je gustoća toka
$$\phi = \frac{v_a T/L}{AT} = \frac{v_a}{V} \xrightarrow[\text{za } V=1]{} v_a$$

tj. $\phi = v_a + v_b$ ako se i čestica b giba (sudarivač).

$$\Rightarrow \quad \sigma = \frac{\Gamma_{fi}}{v_a+v_b} = \frac{1}{4E_aE_b(v_a+v_b)}\int |\mathcal{M}|^2 \, dLips_2$$

Lorentz-invarijantna gustoća toka:

$$F \equiv 4E_aE_b(v_a+v_b) = (\ldots \text{vježbe} \ldots)\, 4\sqrt{(p_a\cdot p_b)^2 - m_a^2 m_b^2}$$

pa je σ isto Lorentz-invarijantna veličina.
Stoga je možemo računati u bilo kojem sustavu

Raspršenje $a+b \to 1+2$ u sustavu središta impulsa (CM)

$$\vec{p}_a = -\vec{p}_b \equiv \vec{p}_i \qquad \vec{p}_1 = -\vec{p}_2 \equiv \vec{p}_f \qquad \left\{ \begin{array}{l} E = mc^2\gamma \\ \vec{p} = m\vec{v}\gamma \end{array}\right.$$

$$F = 4E_aE_b(v_a+v_b) = 4E_aE_b\left(\frac{|\vec{p}_a|}{E_a} + \frac{|\vec{p}_b|}{E_b}\right) = 4\cancel{E_aE_b}\,|\vec{p}_i|\,\frac{E_a+E_b}{\cancel{E_aE_b}}$$

$$\left\{ s = (p_a+p_b)^2 = \left((E_a,\vec{p}_a)+(E_b,\vec{p}_b)\right)^2 = (E_a+E_b, \vec{0})^2 = (E_a+E_b)^2 \right.$$

$$F = 4\sqrt{s}\,|\vec{p}_i|$$

Imali smo za raspad

$$\Gamma = \frac{1}{2m_a}\int |\mathcal{M}|^2 dLips_2 = \frac{|\vec{p}_f|}{32\pi^2 E_a^2}\int |\mathcal{M}_{fi}|^2 d\Omega$$

$E_a^2 \to s$

$\to E_a = \sqrt{s}$ za opći sustav

Znači uz zamjenu $\frac{1}{2\sqrt{s}} \to \frac{1}{F} = \frac{1}{4\sqrt{s}\,|\vec{p}_i|}$ imamo odmah

$$\boxed{\sigma(a+b\to 1+2) = \frac{1}{64\pi^2 s}\frac{|\vec{p}_f|}{|\vec{p}_i|}\int |\mathcal{M}_{fi}|^2 d\Omega}$$

Nekad nas zanima diferencijalni udarni presjek za raspršenje u nekom konkretnom smjeru (ϑ,φ):

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\frac{|\vec{p}_f|}{|\vec{p}_i|}|\mathcal{M}_{fi}|^2}$$

Udarni presjek opisuje raspršenje dviju čestica. Gustoće snopa i mete se opisuju luminozitetom $\mathcal{L}$ i integriranim luminozitetom $\int \mathcal{L}(t)dt$ definiranim tako da je ukupan broj raspršenja u eksperimentu $= \sigma\cdot\int\mathcal{L}\,dt$

$\Rightarrow \left[\int\mathcal{L}\,dt\right] = L^{-2}$ (Npr. ATLAS i CMS 2015-16 su skupili $\approx 15\ fb^{-1}$)

Diracova jednadžba

„Izvod" Schrödingerove jednadžbe za slobodnu česticu:
Izraz za NR energiju

$$E = \frac{\vec{p}^{\,2}}{2m}$$

pretvorimo u operatorsku jednadžbu koja djeluje na jednočestičnu valnu funkciju $\psi(\vec{x},t)$ zamjenama

$$E \to i\frac{\partial}{\partial t} \quad ; \quad \vec{p} \to -i\vec{\nabla}$$

$$\boxed{i\frac{\partial}{\partial t}\psi(\vec{x},t) = -\frac{\vec{\nabla}^2}{2m}\psi(\vec{x},t)}$$ Schrödingerova j.

Gustoću vjerojatnosti i odgovarajuću jednadžbu kontinuiteta dobijemo upotrebom ove jednadžbe

$$\psi^*\cdot(\text{Sch. j.}) - (\text{Sch. j.})^*\psi$$

$$\underbrace{\psi^*\left(i\frac{\partial}{\partial t}\psi\right) - \left(-i\frac{\partial}{\partial t}\psi^*\right)\psi}_{} = (\ldots\vec{\nabla}^2\ldots)$$

$$i\frac{\partial}{\partial t}\underbrace{(\psi^*\psi)}_{\equiv\rho\text{ – gustoća vjerojatnosti } >0} = -i\vec{\nabla}\cdot\vec{j} \longrightarrow \boxed{\frac{\partial}{\partial t}\rho + \vec{\nabla}\cdot\vec{j} = 0}$$

Prirodan pokušaj relativističkog poopćenja je upotreba Einsteinovog izraza za energiju:

$$E^2 = \vec{p}^{\,2} + m^2$$

$$\left(i\frac{\partial}{\partial t}\right)^2\psi = (-i\vec{\nabla})^2\psi + m^2\psi$$

$$\Big(\underbrace{\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2}_{=\partial_\mu\partial^\mu} + m^2\Big)\psi(\underbrace{\vec{x},t}_{x}) = 0$$

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \vec{\nabla}\right)$$

$$\boxed{(\partial_\mu\partial^\mu + m^2)\psi(x) = 0}$$ Klein–Gordonova jednadžba

Rješenje: $\psi(x) = N e^{-ip\cdot x} = N e^{-i(Et - \vec{p}\cdot\vec{x})}$. Provjera:

$$\partial^\mu \psi(x) = \frac{\partial}{\partial x_\mu} N e^{-ip^\mu x_\mu} = (-ip^\mu)\,\psi(x)$$

$$\partial_\mu \partial^\mu \psi(x) = \frac{\partial}{\partial x^\mu}(-ip^\mu) N e^{-ip_\mu x^\mu} = (-ip^\mu)(-ip_\mu)\,\psi(x) = -p^2\psi(x)$$

$$(-p^2 + m^2)\,\psi(x) = (-E^2 + \vec{p}^{\,2} + m^2)\,\psi = 0 \quad \checkmark$$

Usput, relativistički oblik one druge formule je $p^\mu \to i\partial^\mu$.

Dva problema:

1. problem: $\psi_+ = N e^{-i(\overbrace{\sqrt{\vec{p}^{\,2}+m^2}}_{E>0}\,t - \vec{p}\cdot\vec{x})}$ i $\psi_- = N e^{-i(\underbrace{-\sqrt{\vec{p}^{\,2}+m^2}}_{E<0}\,t - \vec{p}\cdot\vec{x})}$

su rješenja i moramo ih uzeti u obzir da bi imali kompletni skup stanja. No što to znači $E<0$ za slobodnu česticu?

2. problem: Gustoća i jednadžba kontinuiteta slijede iz

$$\psi^*(KG\ jj.) - (KG\ jj.)^*\psi$$

$$\psi^*\left(\frac{\partial^2}{\partial t^2}\psi\right) - \left(\frac{\partial^2}{\partial t^2}\psi^*\right)\psi = (\ldots\ \vec{\nabla}^2 \ldots)$$

$$-i\frac{\partial}{\partial t}\underbrace{\left(\psi^* i\frac{\partial}{\partial t}\psi - \left(i\frac{\partial}{\partial t}\psi^*\right)\psi\right)}_{=\rho} = i\vec{\nabla}\cdot\underbrace{(\ldots)}_{=\vec{j}}$$

$=\rho \rightsquigarrow$ može biti <0 jer su ψ i $\left(\frac{\partial}{\partial t}\psi\right)$ nezavisni

$\rho<0$ je gore od $E<0$ jer bi ρ trebala biti gustoća vjerojatnosti.

Uvrštavanjem rješenja $\psi(x) = N e^{-ip\cdot x} = N e^{-i(Et - \vec{p}\cdot\vec{x})}$

$$\rho = N^* \cancel{e^{+ip\cdot x}}(i)(-iE)N\cancel{e^{-ip\cdot x}} - (i)(+iE)|N|^2 = 2|N|^2 E$$

$\vec{j} = (\ldots) = 2|N|^2\vec{p}$ kako bi $j^\mu = (\rho, \vec{j})$ bio 4-vektor

Mogli smo zaključiti $j^\mu \propto p^\mu$ jer je p^μ jedini 4-vektor kojeg imamo na raspolaganju pa odmah vidimo $\rho \propto E$.

Vidimo da su dva problema povezana.

Dirac napada 2. problem tako da linearizira jednadžbu gibanja u $\frac{\partial}{\partial t}$ (vidi se da u Klein-Gordon slučaju $\frac{\partial^2}{\partial t^2}$ dovodi do problema)

Diracov „Ansatz":

$$\underbrace{(\vec{\alpha}\cdot\vec{p} + \beta m)}_{\text{Diracov hamiltonijan}}\psi = E\psi$$

gdje ψ i dalje mora zadovoljavati KG jednadžbu, a $\vec{\alpha}$ i β su konstante koje to trebaju omogućiti.

Umjesto $\vec{\alpha}$ i β uvodimo ekvivalentne $\vec{\gamma}$ i γ^0 tako da je

$$\vec{\alpha} = \gamma^{0^{-1}}\vec{\gamma} \quad ; \quad \beta = \gamma^{0^{-1}}$$

(vidjet ćemo kasnije da je $\gamma^{0^{-1}} = \gamma^0$)

Tako Diracov Ansatz postaje

$$\gamma^0 / \quad \gamma^{0^{-1}}\vec{\gamma}\cdot\vec{p}\,\psi + \gamma^{0^{-1}} m\psi = E\psi \qquad ; \ \vec{p} \to -i\vec{\nabla}\,,\ E \to i\frac{\partial}{\partial t}$$

$$\gamma^0\left(i\frac{\partial}{\partial t}\right)\psi - \vec{\gamma}\cdot(-i\vec{\nabla})\psi - m\psi = 0$$

$$\gamma^\mu = (\gamma^0, \vec{\gamma}) \quad , \quad \partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \vec{\nabla}\right)$$

$$(i\gamma^\mu\partial_\mu - m)\psi = 0$$

⇐ ovo je sad manifestno relativistički kovarijantno

Koja svojstva moraju imati γ^μ da bi $\psi(x)$ zadovoljavala KG jednadžbu?

$$(i\gamma^\nu\partial_\nu + m) \,/\quad (i\gamma^\mu\partial_\mu - m)\psi = 0$$

$$\left[-\gamma^\nu\partial_\nu\gamma^\mu\partial_\mu + im\underbrace{(\gamma^\mu\partial_\mu - \gamma^\nu\partial_\nu)}_{=0} - m^2\right]\psi = 0$$

$-\gamma^\nu\partial_\nu\gamma^\mu\partial_\mu$ ↓ mora biti $= \partial^\mu\partial_\mu$ da bi imali $(\partial^\mu\partial_\mu + m^2)\psi = 0$

Kako su γ^μ konstante:

$$\gamma^\nu \partial_\nu \gamma^\mu \partial_\mu = \gamma^\nu \gamma^\mu \partial_\nu \partial_\mu \stackrel{!}{=} \partial^\mu \partial_\mu \qquad \left\{ \begin{array}{l} \mu,\nu = 0,1,2,3 \\ i,j = 1,2,3 \end{array} \right.$$

$$\Rightarrow \gamma^0\gamma^0\partial_0\partial_0 + \underbrace{\gamma^0\gamma^j\partial_0\partial_j + \gamma^j\gamma^0\partial_j\partial_0}_{=0} + \gamma^i\gamma^j\partial_i\partial_j = \partial_0\partial_0 - \partial_i\partial_i$$

$$\Rightarrow (\gamma^0)^2 = 1 \quad \text{tj.} \quad (\gamma^0)^{-1} = \gamma^0 \qquad (1)$$

$$\Rightarrow \gamma^0\gamma^i + \gamma^i\gamma^0 = 0 \qquad (2)$$

$$(i,j)\text{-dio:} \quad \left(\gamma^1\gamma^1\partial_1\partial_1 + \gamma^2\gamma^2\partial_2\partial_2 + \gamma^3\gamma^3\partial_3\partial_3\right) + \underbrace{\sum_{i\neq j}\gamma^i\gamma^j\partial_i\partial_j}_{} = -\partial_1\partial_1 - \partial_2\partial_2 - \partial_3\partial_3$$

$$\Rightarrow (\gamma^i)^2 = -1 \qquad i=1,2,3 \qquad (3)$$

$$\underbrace{\gamma^1\gamma^2\partial_1\partial_2 + \gamma^2\gamma^1\partial_2\partial_1}_{(\gamma^1\gamma^2+\gamma^2\gamma^1)\partial_1\partial_2} = 0, \qquad \underbrace{(\gamma^1\gamma^2+\gamma^2\gamma^1)}_{=0}$$

$$\Rightarrow \gamma^i\gamma^j + \gamma^j\gamma^i = 0 \qquad i \neq j \qquad (4)$$

Analiziramo (1)–(4)

$\gamma^0\gamma^i = -\gamma^i\gamma^0 \Rightarrow$ ne komutiraju jer bi $\gamma^0\gamma^i \stackrel{(2)}{=} -\gamma^i\gamma^0 \stackrel{\text{komut.}}{=} -\gamma^0\gamma^i = 0$ pa Diracov ansatz ne bi imao smisla. $(\alpha_i = 0)$

γ^μ moraju biti nekomutativni objekti $\Rightarrow$ matrice

(1)–(4) u združenom, relativistički kovarijantnom obliku:

$$\boxed{\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu \equiv \underbrace{\{\gamma^\mu, \gamma^\nu\}}_{\text{antikomutator}} = 2g^{\mu\nu}}$$

Diracove gama matrice

Uz takve γ^μ, $\psi(x)$ zadovoljava relativistički uvjet i imamo

$$\boxed{(i\gamma^\mu\partial_\mu - m)\psi(x) = 0}$$

Diracova jednadžba

Diracov hamiltonijan mora biti hermitski $H^\dagger = H = \gamma^0\vec{\gamma}\cdot\vec{p} + \gamma^0 m$

$\Rightarrow \gamma^{0\dagger} = \gamma^0$; $\underbrace{(\gamma^0\gamma^i)^\dagger}_{\gamma^{i\dagger}\gamma^{0\dagger}} = \gamma^0\gamma^i \; / \gamma^0 \Rightarrow \gamma^{i\dagger} = \gamma^0\gamma^i\gamma^0 = -\underbrace{\gamma^0\gamma^0}_{=1}\gamma^i = -\gamma^i$

združeno: $\boxed{\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0}$

Sva druga bitna svojstva γ-matrica slijede iz antikomutacijskih relacija.

Npr.

cikličnost traga

$$\mathrm{Tr}\,\gamma^i = \mathrm{Tr}\,\gamma^i\gamma^0\gamma^0 \overset{\downarrow}{=} \mathrm{Tr}\,\gamma^0\gamma^i\gamma^0 = -\mathrm{Tr}\,\gamma^i\gamma^0\gamma^0 = 0$$

$$\mathrm{Tr}\,\gamma^0 = \mathrm{Tr}\,\gamma^0\gamma^1\gamma^1 = (\ldots) = 0$$

$$\Rightarrow \boxed{\mathrm{Tr}\,\gamma^\mu = 0}$$

(Za druga svojstva vidi vježbe i literaturu.)

Da bi zadovoljile sva potrebna svojstva matrice trebaju biti barem 4×4.

Popularan izbor za eksplicitni oblik γ-matrica je od Diraca i Paulija

$$\gamma^0 = \begin{pmatrix} \mathbb{1}_{2\times2} & 0 \\ 0 & -\mathbb{1}_{2\times2} \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

gdje su σ^i Paulijeve matrice, $\sigma^{i\dagger} = \sigma^i$

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Dakle,
$$\left(i\gamma^\mu_{4\times4}\partial_\mu - m\mathbb{1}_{4\times4}\right)\psi(x) = 0$$

$\Rightarrow \psi(x)$ je zapravo $\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$ – Diracov spinor

veza sa spinom će postati jasna kasnije

Koje je značenje te četiri komponente?

Prije interpretacije $\psi_{1,\dots,4}$, uvjerimo se da smo riješili problem negativne gustoće vjerojatnosti rješenja KG jednadžbe:

$$\psi^+ \gamma^0 \,(\text{Diracova j.}) - (\text{Diracova j.})^+ \gamma^0 \psi = 0$$

ubacujemo da bi se riješili ovog

$$(i\gamma^0\partial_0 + i\gamma^i\partial_i - m)\psi = 0$$

$$\psi^+\gamma^0(i\gamma^0\partial_0 + i\gamma^i\partial_i - \cancel{m})\psi - \psi^+(-i\underbrace{\gamma^{0+}}_{\gamma^0}\overleftarrow{\partial_0} - i\underbrace{\gamma^{i+}}_{-\gamma^i}\overleftarrow{\partial_i} - \cancel{m})\gamma^0\psi = 0 \quad / \cdot (-i)$$

$$\psi^+\partial_0\psi + (\partial_0\psi^+)\psi + \psi^+\gamma^0\gamma^i\partial_i\psi - (\partial_i\psi^+)\underbrace{\gamma^i\gamma^0}_{-\gamma^0\gamma^i}\psi = 0$$

$$\partial_0\underbrace{(\psi^+\psi)}_{\rho>0\,!} + \partial_i\underbrace{(\psi^+\gamma^0\gamma^i\psi)}_{j^i} = 0 \quad \Rightarrow \quad \frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0 \;\checkmark$$

gustoća struje („struja"): $j^\mu = (\rho, j^i) = \psi^+\gamma^0\gamma^\mu\psi$

$$\boxed{\partial_\mu j^\mu = 0}$$

kovarijantni oblik jednadžbe kontinuiteta, sačuvanje struje

Det. $\bar{\psi} \equiv \psi^+\psi^0$ – adjungirani spinor

pa je struja $\boxed{j^\mu = \bar{\psi}\gamma^\mu\psi}$.

Spin i Diracova jednadžba

U QM očuvane veličine su one čiji odgovarajući operatori komutiraju s hamiltonijanom

$$\frac{d}{dt}O = [H, O] = 0$$

Npr. $\vec{L} = \vec{r} \times \vec{p}$ je očuvan jer $[H, \vec{L}] = 0$.

za Diracov hamiltonijan međutim imamo

$$[H, L^i] = [\underbrace{\gamma^0\gamma^l}_{\text{komutira s } 1\!\!1} p^l + \underbrace{\gamma^0 m}_{0 \text{ (komutira)}}, \varepsilon^{ijk} r^j \underbrace{p^k}_{\text{komutira s } p_l} 1\!\!1_{4\times4}] = \gamma^0\gamma^l \varepsilon^{ijk} \underbrace{[p^l, r^j]}_{-i\delta^{lj}} p^k = \underbrace{-i\gamma^0 \varepsilon^{ijk} \gamma^j p^k}_{i(\gamma^0\vec{\gamma}\times\vec{p})^i} \neq 0$$

Promotrimo sada operator $\vec{S} \equiv \frac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$

$$[H, S^i] = [\gamma^0\gamma^l p^l + \underbrace{\gamma^0 m}_{\begin{pmatrix} 1\!\!1 & 0 \\ 0 & -1\!\!1 \end{pmatrix} \to \text{komutira}}, \frac{1}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}] = \frac{1}{2} p^l \gamma^0 \left[\begin{pmatrix} 0 & -\sigma^l \\ \sigma^l & 0 \end{pmatrix}, \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}\right]$$

$$= \frac{1}{2} p^l \gamma^0 \left\{ \begin{pmatrix} 0 & -\sigma^l\sigma^i \\ \sigma^l\sigma^i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -\sigma^i\sigma^l \\ \sigma^i\sigma^l & 0 \end{pmatrix} \right\} = \frac{1}{2} p^l \gamma^0 \begin{pmatrix} 0 & -[\sigma^l, \sigma^i] \\ \underbrace{[\sigma^l, \sigma^i]}_{2i\underbrace{\varepsilon^{lij}}_{=\varepsilon^{ijl}}\sigma^j} & 0 \end{pmatrix} = i\varepsilon^{ijl}\gamma^0\gamma^j p^l = i(\gamma^0\vec{\gamma}\times\vec{p})^i$$

$$\Rightarrow [H, L^i + S^i] = 0$$

tj. $\vec{L} + \vec{S}$ je očuvana veličina. $\vec{S} \equiv$ spin

$\vec{S}\psi$: $\vec{S}$ djeluje na „interne" komponente $\psi_{1\ldots4}$ spinora (Sad razumijemo zašto se ψ zove „spinor")

Svojstvene vrijednosti od $\vec{S}$?

QM: $\vec{S}^2|\psi\rangle = s(s+1) 1\!\!1 |\psi\rangle$

$$\vec{S}^2 = \frac{1}{4}\begin{pmatrix} \vec{\sigma}^2 & 0 \\ 0 & \vec{\sigma}^2 \end{pmatrix} = \left\{ \begin{array}{l} \vec{\sigma}^2 = \sigma^i\sigma^i \\ \sigma^i\sigma^j = \delta^{ij} + i\varepsilon^{ijk}\sigma^k \quad /\delta^{ij} \\ \sigma^i\sigma^i = \delta^{ii} + 0 = 3 \end{array} \right\} = \frac{3}{4}\begin{pmatrix} 1\!\!1 & 0 \\ 0 & 1\!\!1 \end{pmatrix} \Rightarrow \boxed{s = \frac{1}{2}}$$

tj. Diracova jednadžba opisuje čestice spina $\frac{1}{2}$.

U QM spinor za čestice spina $\frac{1}{2}$ ima samo 2 komponente $\begin{pmatrix} |s=\frac{1}{2}, m=\frac{1}{2}\rangle \\ |s=\frac{1}{2}, m=-\frac{1}{2}\rangle \end{pmatrix}$

Zašto Diracov spinor ima 4 komponente ?

Rješenje Diracove jednadžbe (za slobodan čestica)

Prirodni „ansatz" je ravni val:

$$\psi(x) = u(E,\vec{p})\, e^{-ip\cdot x}$$

$u(E,\vec{p})$ – isto se naziva Diracov spinor, stupac s 4 komponente

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = \left(i\gamma^\mu \frac{\partial}{\partial x^\mu} - m\right) u(E,\vec{p})\, e^{-ip_\mu x^\mu}$$

$$= \underbrace{\left[i\gamma^\mu(-ip_\mu) - m\right] u(E,\vec{p})}_{=0}\, \underbrace{e^{-ip\cdot x}}_{\neq 0} = 0$$

$$\boxed{(\underbrace{\gamma^\mu p_\mu}_{E\gamma^0 + p_i\gamma^i = E\gamma^0 - p^i\gamma^i} - m)\, u(E,\vec{p}) = 0}$$

Diracova jednadžba za spinor $u(E,\vec{p})$

$$\left[E\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}\cdot\vec{p} - m\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\begin{pmatrix} u_A \\ u_B \end{pmatrix} = 0$$

$u_{A,B}$ – dvokomponentni spinori

$$\begin{pmatrix} (E-m)\mathbb{1} & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -(E+m)\mathbb{1} \end{pmatrix}\begin{pmatrix} u_A \\ u_B \end{pmatrix} = 0$$

$$(E-m)u_A - \vec{\sigma}\cdot\vec{p}\, u_B = 0$$

$$\vec{\sigma}\cdot\vec{p}\, u_A - (E+m)u_B = 0 \implies u_B = \frac{\vec{\sigma}\cdot\vec{p}}{E+m} u_A$$

Za bilo koji u_A i $u_B = \dfrac{\vec{\sigma}\cdot\vec{p}}{E+m} u_A \ldots$

$$\{(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = \vec{a}\cdot\vec{b} + i(\vec{a}\times\vec{b})\cdot\vec{\sigma}$$

$$(E-m)u_A - \frac{(\vec{\sigma}\cdot\vec{p})(\vec{\sigma}\cdot\vec{p})}{E+m} u_A = (E-m)u_A - \frac{\vec{p}^{\,2}}{E+m} u_A =$$

$$= (E-m)u_A - \frac{E^2 - m^2}{E+m} u_A = \left[E - m - \frac{(E-m)\cancel{(E+m)}}{\cancel{E+m}}\right] u_A = 0$$

... pa imamo rješenje Diracove jednadžbe

$$u(E,\vec{p}) = \begin{pmatrix} u_A \\ \dfrac{\vec{\sigma}\cdot\vec{p}}{E+m} u_A \end{pmatrix}$$

Biramo dva linearno nezavisna rješenja definirana s

$$u_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{i} \quad u_A = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

tj.

$$u_1(E,\vec{p}) = N_1 \begin{pmatrix} 1 \\ 0 \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} \quad ; \quad u_2(E,\vec{p}) = N_2 \begin{pmatrix} 0 \\ 1 \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix}$$

$N_{1,2}$ – normalizacija koju ćemo odrediti kasnije

Preostala dva rješenja dobivamo uvrštavanjem bilo kojeg u_B u prvu gornju jednadžbu $u_A = \frac{\vec{\sigma}\cdot\vec{p}}{E-m} u_B$

što na isti način vodi na

$$u_3(E,\vec{p}) = N_3 \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E-m}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ 1 \\ 0 \end{pmatrix} \quad ; \quad u_4(E,\vec{p}) = N_4 \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E-m}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ 0 \\ 1 \end{pmatrix}$$

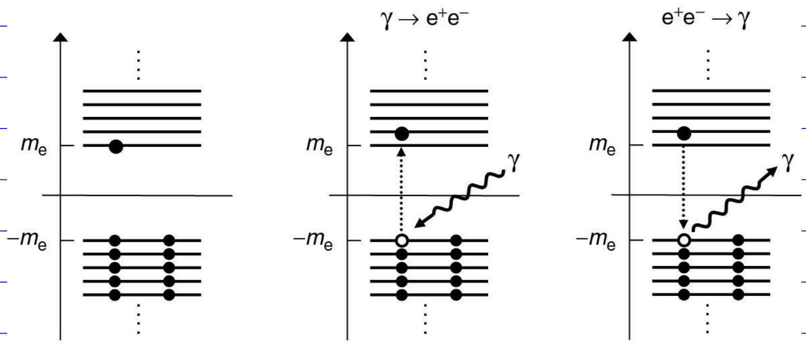
Pogledajmo ova rješenja u sustavu mirovanja čestice: $\vec{p}=0$, $E=m$

$$(\gamma^0 E - m)u = \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix}\begin{pmatrix} u_A \\ u_B \end{pmatrix} = m \begin{pmatrix} u_A \\ u_B \end{pmatrix}$$

Za $u_{1,2}$: $\begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix}\begin{pmatrix} u_A \\ 0 \end{pmatrix} = m \begin{pmatrix} u_A \\ 0 \end{pmatrix} \rightarrow E u_A = m u_A$

Za $u_{3,4}$: $\begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix}\begin{pmatrix} 0 \\ u_B \end{pmatrix} = m \begin{pmatrix} 0 \\ u_B \end{pmatrix} \Rightarrow E u_B = -m u_B$ $\qquad E < 0$?!

Problem negativnih energija Dirac rješava oslanjanjem na Paulijev princip i postuliranjem beskonačnog mora popunjenih stanja negativne energije



→ otkriće pozitrona (Anderson, 1933)

→ svaka čestica ima antičesticu iste mase i suprotnog naboja (vrijedi i za bozone)

Alternativna interpretacija rješenja s $E<0$ (Feynman i Stückelberg):
Ona se zapravo gibaju unatrag u vremenu što je matematički ekvivalentno antičestici koja se giba unaprijed u vremenu.

$$e^{-i\underset{<0}{E}(-t)} = e^{-i|E|t}$$

„Gibanje" je ionako nespretan koncept za FEČ. Iz obje interpretacije je dovoljno prihvatiti ekvivalenciju:

$$\begin{pmatrix}\text{apsorpcija } E<0 \text{ elektrona}\\ \text{impulsa } -\vec{p} \text{ i spina } -S\end{pmatrix} \Longleftrightarrow \begin{pmatrix}\text{emisija } E>0 \text{ pozitrona}\\ \text{impulsa } +\vec{p} \text{ i spina } +S\end{pmatrix}$$

i obratno.

Motivirani time gledamo $E<0$, $-\vec{p}$ spinore $u_{3,4}$:

$$u_{3,4}(E,-\vec{p})\, e^{-i(Et+\vec{p}\cdot\vec{x})} = \underbrace{u_{3,4}(-|E|,-\vec{p})}_{\boxed{=v_{2,1}(|E|,\vec{p})}}\ \underbrace{e^{-i(-|E|t+\vec{p}\cdot\vec{x})}}_{e^{+ip\cdot x}\ \text{uz}\ p^\mu=(|E|,\vec{p})}$$

antičestični spinori $\quad$ ($3,4 \to 2,1$ zbog $S\to-\vec{S}$)

Pa su druga dva rješenja Diracove jednadžbe $v_{1,2}\,e^{+ip\cdot x}$

$$v_1(E,\vec{p}) = u_4(-E,-\vec{p}) = N_4\begin{pmatrix}\frac{\vec{\sigma}\cdot(-\vec{p})}{-E-m}\begin{pmatrix}1\\0\end{pmatrix}\\ \begin{pmatrix}1\\0\end{pmatrix}\end{pmatrix} = N_4\begin{pmatrix}\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\begin{pmatrix}1\\0\end{pmatrix}\\ \begin{pmatrix}1\\0\end{pmatrix}\end{pmatrix}$$

$$v_2(E,\vec{p}) = u_3(-E,-\vec{p}) = N_3\begin{pmatrix}\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\begin{pmatrix}0\\1\end{pmatrix}\\ \begin{pmatrix}0\\1\end{pmatrix}\end{pmatrix}$$

i ovdje i uvijek odsada je svugdje $E>0$!

Normalizacija: $\int d^3x\,\rho = \int d^3x\,\psi^+\psi = 2E$ (relativistički kovarijantno)

$$\int d^3x\,(u_1\cancel{e^{-ip\cdot x}})^+ u_1\cancel{e^{-ip\cdot x}} = \underset{=1}{V}\,u_1^+u_1 = |N_1|^2\left(u_A^+,\ u_A^+\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\right)\begin{pmatrix}u_A\\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}u_A\end{pmatrix} =$$

$$= |N_1|^2\Big(\underbrace{u_A^+u_A}_{=1} + u_A^+\underbrace{\frac{(\vec{\sigma}\cdot\vec{p})(\vec{\sigma}\cdot\vec{p})}{(E+m)^2}}_{\frac{1}{(E+m)^2}\cdot\vec{p}^{\,2} = \frac{E^2-m^2}{(E+m)^2} = \frac{E-m}{E+m}}u_A\Big) = |N_1|^2\left(1+\frac{E-m}{E+m}\right) = |N_1|^2\frac{E+\cancel{m}+E-\cancel{m}}{E+m}$$

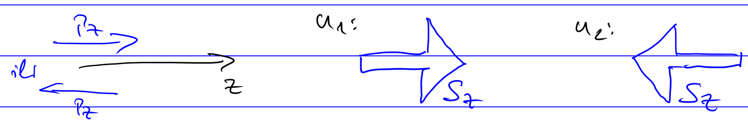
$$= |N_1|^2\frac{2E}{E+m} \underset{\uparrow\ \text{normalizacija}}{=} 2E \quad\Rightarrow\quad N_1 = \sqrt{E+m} = N_2 = N_3 = N_4$$

Spin, helicitet i kiralnost

Vidjeli smo da je operator spina $\vec{S} = \frac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$.
QM dopušta samo jednu komponentu kao značajku čestice u kojem trenutku. Obično se bira S_z.

Čestica koja se giba duž z-osi: $\vec{p} = p_z \hat{z}$.

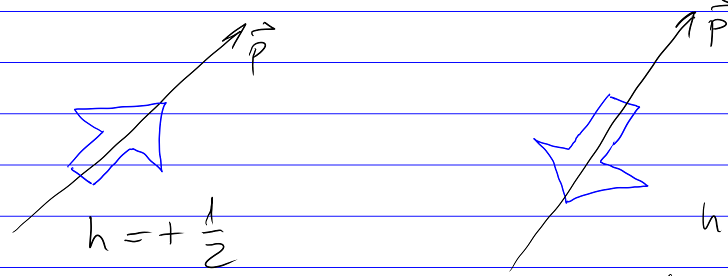
$$S_z \psi(x) = \frac{1}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} N \begin{pmatrix} u_A \\ \frac{\sigma_3 p_z}{E+m} u_A \end{pmatrix} = \left\{ \begin{array}{l} \sigma_3 u_A = \pm u_A \\ u_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ ili } \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{array} \right\} = \pm \frac{1}{2} \psi(x)$$



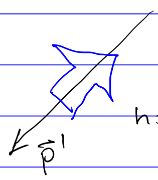
ali čestice koje se ne gibaju duž z-osi nemaju definiran S_z
jer $[\sigma_3, \sigma_{1,2}] \neq 0$. Operator spina je problematičan: zbog $[H, \vec{S}] \neq 0$.

Stoga uvodimo helicitet $\boxed{h \equiv \vec{S} \cdot \hat{p}}$ (spin duž impulsa čestice)

$[H, \vec{S} \cdot \hat{p}] = 0$ (vježbe)



Dovoljno veliki Lorentzov potisak okreće $\vec{p}$, ali ne i S



$h = -\frac{1}{2}$ ⇒ helicitet nije invarijantan na Lorentzove transformacije! ($\vec{S} \cdot \hat{p}$ je očito neinvarijantno kao produkt 3-vektora)

Treća veličina vezana uz „vrtnju" elektrona:

kiralnost: $\boxed{\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^4}$

U Dirac-Pauli reprezentaciji $\gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$\{\gamma^5, \gamma^\mu\} = 0$$

pa γ^5 komutira s operatorom Lorentzove transformacije.
(vidi ili App. B.2 (Thomson) ili $S = \exp\left(-\frac{i}{4}[\gamma^\mu, \gamma^\nu]\right)$ - vježbe?)

i kiralnost je invarijantna. No $[H, \gamma^5] \neq 0$ općenito što nije takav problem jer u ultrarelativističkom limitu $E \gg m$ imamo npr.

$$\psi_1 = N\begin{pmatrix} u_A \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m} u_A \end{pmatrix} \approx N \begin{pmatrix} u_A \\ \vec{\sigma}\cdot\hat{p}\, u_A \end{pmatrix} \Rightarrow \gamma^5\psi_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\psi_1 = N\begin{pmatrix} \vec{\sigma}\cdot\hat{p}\, u_A \\ u_A \end{pmatrix} = N\begin{pmatrix} \vec{\sigma}\cdot\hat{p} & 0 \\ 0 & \vec{\sigma}\cdot\hat{p} \end{pmatrix}\begin{pmatrix} u_A \\ \vec{\sigma}\cdot\hat{p}\, u_A \end{pmatrix}$$

$$= 2\vec{S}\cdot\hat{p}\,\psi_1 \qquad \text{jer } (\vec{\sigma}\cdot\hat{p})^2 = \hat{p}^2 = 1$$

pa je u tom limitu $\frac{1}{2}$ kiralnost = helicitet : često za čestice kiralnosti +1 ili -1 govori da su „desne" i „lijeve".

$$\left([H(m=0), \gamma^5] = [\gamma^0\vec{\gamma}\cdot\vec{p}, \gamma^5] = 0 \right)$$

Međudjelovanje izmjenom čestice

U NR QM vremenski ovisan račun smetnje

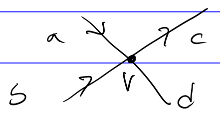
$$T_{fi} = \langle f|V|i\rangle + \sum_{j\neq i} \frac{\langle f|V|j\rangle\langle j|V|i\rangle}{E_i - E_j}$$

V – statični potencijal – nepodoban za relativističku fiziku
Npr. u raspršenju druge čestice je izvor potencijala i mora biti sposobna primiti dio impulsa.

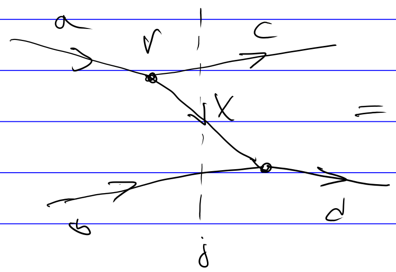
QFT: Međustanje $|j\rangle$ sadrži novu česticu koja se propagira između dvije točke djelovanja potencijala i prenosi impuls.

Npr. za raspršenje $a+b \to c+d$

$\langle cd|V|ab\rangle = 0$ u većini slučajeva (većina SM interakcija uključuje točno 3 čestice)



$= 0$



$$= \frac{\langle d|V|bX\rangle\langle cX|V|a\rangle}{(E_a+E_b)-(E_b+E_X+E_c)}$$

Stvaranje i poništavanje čestica nisu predviđeni Diracovom jednočestičnom valnom funkcijom $\psi(x)$ koja poštuje očuvanje vjerojatnosti. Stoga je cijeli sljedeći „izvod" Feynmanove amplitude i kasnije Feynmanovih dijagrama neegzaktan i oslonjen na intuiciju. (Za stroži izvod vidi zborne kolegije QFT I i II)

Relativistička normalizacija je vodila na invarijantnu amplitudu

$$\mathcal{M}_{cd,ab} = \sqrt{(2E_a)(2E_b)(2E_c)(2E_d)}\; T_{cd,ab}$$

pa ćemo imati

$$\mathcal{M}_{cd,ab} \equiv \mathcal{M} = \frac{1}{2E_x}\, \frac{\langle d|V|bx\rangle\langle cx|V|a\rangle}{(E_a+\cancel{E_b})-(E_c+E_x+\cancel{E_b})}$$

ova stanja su dobivena zamjenom $|a\rangle \to \frac{1}{\sqrt{2E_a}}|a\rangle \ldots$ izrazu za T i jedno $\left(\frac{1}{\sqrt{2E_x}}\right)^2$ se ne pokrati pri prelasku na $\mathcal{M}$.

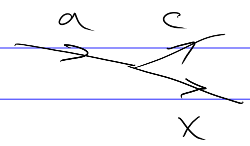
Da bi $\mathcal{M}$ bilo Lorentz-invarijantno $\langle cx|V|a\rangle$ mora imati dobra transformacijska svojstva. Pretpostavimo da je to skalar

$$\langle cx|V|a\rangle \equiv g_a$$
$$\langle d|V|bx\rangle \equiv g_b$$

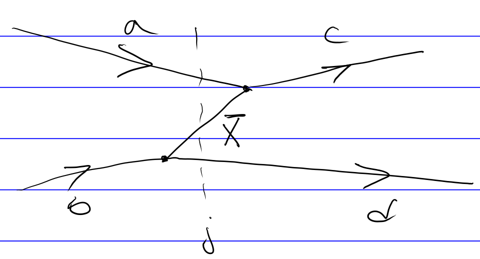
jakost interakcije (broj)

$$\mathcal{M} = \frac{g_a g_b}{(2E_x)(E_a-E_c-E_x)}$$

Treba uočiti da $E_x \neq E_a - E_c$ tj. u „interakciji" nije očuvana energija. Ali 3-impuls jest.



No, postoji i drugi redoslijed dviju interakcija



(apsorpcija $\bar{X}$ ima isti efekt kao emisija X ranije, ovdje $\bar{X}$ je antičestica od X)

$$= \frac{1}{2E_x}\, \frac{g_a g_b}{(\cancel{E_a}+E_b)-(\cancel{E_a}+E_d+E_x)}$$

$$E_{\bar X} = \vec p_{\bar X}^{\,2} + m_X^2 \,, \quad \vec p_{\bar X} = \vec p_c - \vec p_a = -\vec p_X \Rightarrow E_{\bar X} = E_X$$

$$\mathcal{M} = \frac{g_a g_b}{2E_X}\left(\frac{1}{E_a - E_c - E_X} + \frac{1}{E_b - E_d - E_X}\right)$$

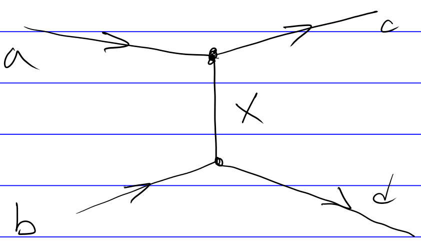
"Vanjska" energija je naravno očuvana: $E_a + E_b = E_c + E_d$

$$\mathcal{M} = \frac{g_a g_b}{2E_X}\,\frac{\cancel{E_a - E_c} + E_X - (\cancel{E_a - E_c} - E_X)}{(E_a - E_c)^2 - E_X^2} = \frac{g_a g_b}{(E_a - E_c)^2 - E_X^2}$$

$$= \frac{g_a g_b}{(E_a - E_c)^2 - (\vec p_a - \vec p_c)^2 - m_X^2} = \frac{g_a g_b}{(p_a - p_c)^2 - m_X^2}$$

Super: dva ponaosob neinvarijantna dijagrama su se zbrojila u invarijantni izraz.

Feynman: zamjenjujemo sve ovo s <u>jednim</u> dijagramom



gdje vremenski uređaj dvaju interakcijskih vrhova nije definiran i gdje nije definirano da li je X čestica ili antičestica.

Izmjena X (anti)čestice je kvantificirana invarijantnim

$$\boxed{\text{Feynmanovim propagatorom} = \frac{1}{q^2 - m_X^2}}$$

gdje je u svakom vrhu 4-vektor impulsa očuvan

$$q^\mu = p_a^\mu - p_c^\mu = p_d^\mu - p_b^\mu$$

ali <u>ne vrijedi</u> $q^2 = m_X^2$!! $\longrightarrow$ VIRTUALNA ČESTICA
- neopservabilna
- matematički konstrukt

($q^2 < 0$ često)

Kvantna elektrodinamika

Interakcija nabijene čestice s em poljem u klasičnoj fizici

„minimalna supstitucija":
$$\left.\begin{array}{l}\vec{p} \longrightarrow \vec{p} - g\vec{A} \\ E \longrightarrow E - g\phi\end{array}\right\} \quad p^\mu \to p^\mu - gA^\mu$$

U QM $p^\mu \to i\partial^\mu$ pa Diracova jednadžba postaje

$$\left[i\gamma^\mu \partial_\mu - g\gamma^\mu A_\mu - m\right]\psi(x) = 0$$

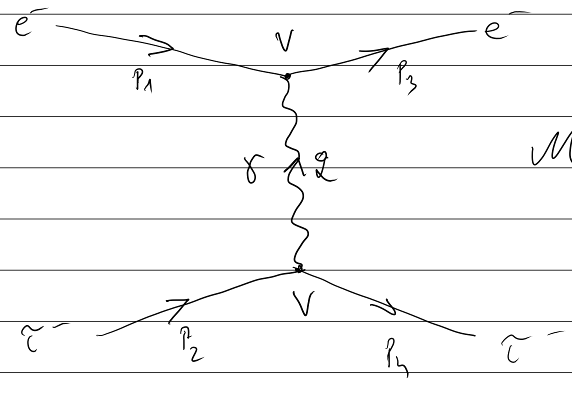
(U FEČ II ćemo mogući minimalne supstitucije zamijeniti dubljom mogućom principa baždarne invarijantnosti.)

Dakle, Diracov hamiltonijan:

$$\gamma^0 / \text{D.j.} \Rightarrow i\frac{\partial}{\partial t}\psi(x) = \Big(\underbrace{-i\gamma^0\vec{\gamma}\cdot\vec{\nabla} + \gamma^0 m}_{H_0} + \underbrace{g\,\gamma^0\gamma^\mu A_\mu}_{V}\Big)\psi(x)$$

Npr. elektrostatika $A^\mu = (\phi, 0) \to V = g\gamma^0\gamma^0\phi = g\phi$ ✓

Računamo raspršenje $e^-\tau^- \to e^-\tau^-$ u QED



$$\mathcal{M} = \langle\psi_3|V|\psi_1\rangle \frac{1}{q^2 - 0^2}\langle\psi_4|V|\psi_2\rangle$$

U kvantnoj teoriji: $\psi_1 = u(p_1, s_1)\, e^{-ip_1\cdot x}$ $\quad s_1 = 1,2$ – spin, helicitet, … zanemarujemo privremeno

$$A^\mu = \varepsilon^\mu(q, \lambda)\, e^{-iq\cdot x}$$
↳ polarizacija fotona

$\lambda = 1,2$ za realni foton (dva polarizacijska stupnja slobode u ravnini transverzalnoj na $\vec{q}$)

$\langle \psi_3 | V | \psi_1 \rangle$ potječe od $\int d^4x \, \psi_3^\dagger(x) \, V(x) \, \psi_1(x) =$

$$= \int d^4x \, \underbrace{u(p_3)^\dagger}_{= u_3^\dagger} e^{+ip_3 \cdot x} \, q \gamma^0 \gamma^\mu \varepsilon_\mu(q,\lambda) e^{-iq\cdot x} \, u_1 e^{-ip_1 \cdot x}$$

$$= \underbrace{(2\pi)^4 \delta^4(p_3 - q - p_1)}_{\text{očuvanje 4-impulsa u svakom vrhu}} \cdot \underbrace{q \underbrace{u_3^\dagger \gamma^0}_{\overline{u_3}} \gamma^\mu u_1}_{= j_e^\mu \text{ - 4-vektor struje}} \cdot \varepsilon_\mu(q,\lambda)$$

↓

ulazi u dLips

$$\boxed{\begin{array}{c} q = Q_e e \\ e > 0 \\ Q_e = -1 \end{array}}$$

Slično $\langle \psi_4 | V | \psi_2 \rangle \longrightarrow e Q_\tau \overline{u_4} \gamma^\mu u_2 \, \varepsilon_\mu^*(q,\lambda)$

emisija iz vrha (trebamo $\delta^4(p_4 + q - p_2)$)

Pa je konačno:

$$\mathcal{M} = e Q_e \overline{u_3} \gamma^\mu u_1 \boxed{\frac{\sum_\lambda \varepsilon_\mu(q,\lambda) \varepsilon_\nu^*(q,\lambda)}{q^2}} e Q_\tau \overline{u_4} \gamma^\nu u_2$$

kompletan propagator fotona

Za <u>realne</u> fotone, transverzalna polarizacija:

$$\varepsilon^\mu(q,1) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \varepsilon^\mu(q,2) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\sum_{\lambda=1,2} \varepsilon^\mu(q,\lambda)^* \varepsilon^\nu(q,\lambda) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Za <u>virtualne</u> fotone, sve 4 polarizacije su moguće i

$$\boxed{\sum_\lambda \varepsilon^\mu(q,\lambda)^* \varepsilon^\nu(q,\lambda) = -g^{\mu\nu}} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

Vidi Thomson, Appendix D za detalje

Doprinosi ovih dviju polarizacija se krate za realne fotone pa i za njih možemo rabiti $\sum_{\lambda=1,2} \varepsilon^{\mu*} \varepsilon^\nu = -g^{\mu\nu}$

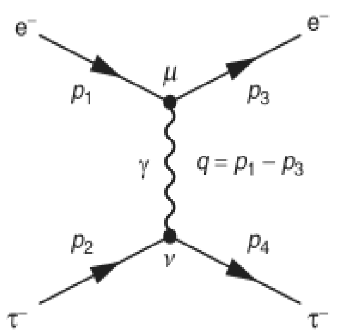
Konačna Feynmanova amplituda

$$\mathcal{M}(e^-\tau^-\to e^-\tau^-) = eQ_e(\bar{u}_3\gamma^\mu u_1)\,\frac{-g_{\mu\nu}}{q^2}\,(\bar{u}_4\gamma_\nu u_2)$$

se može dobiti i izravno, primjenom Feynmanovih pravila za Feynmanove dijagrame:

initial-state particle:	$u(p)$
final-state particle:	$\bar{u}(p)$
initial-state antiparticle:	$\bar{v}(p)$
final-state antiparticle:	$v(p)$
initial-state photon:	$\varepsilon_\mu(p)$
final-state photon:	$\varepsilon^*_\mu(p)$
photon propagator:	$-\frac{ig_{\mu\nu}}{q^2}$
fermion propagator:	$-\frac{i(\gamma^\mu q_\mu + m)}{q^2-m^2}$
QED vertex:	$-iQe\gamma^\mu$

Pravila daju $-i\mathcal{M}$.



$\bar{u}(p_3)[ie\gamma^\mu]u(p_1)$

$\frac{-ig_{\mu\nu}}{q^2}$

$\bar{u}(p_4)[ie\gamma^\nu]u(p_2)$

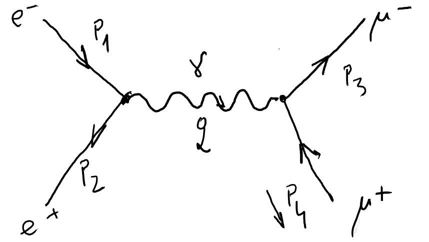
$$\cancel{-i}\mathcal{M} = \bar{u}_3(-\cancel{i}Q_e e\gamma^\mu)u_1\,\frac{-ig_{\mu\nu}}{q^2}\,\bar{u}_4(-iQ_\tau e\gamma_\nu)u_2 \implies$$ isti $\mathcal{M}$ kao ranije.

Fermionski propagator $\propto \frac{\not{q}+m}{q^2-m^2} = \frac{\sum_s u(q,s)\bar{u}(q,s)}{q^2-m^2}$

$\uparrow$ vježbe, relacija kompletnosti

Dakle ista struktura za sve čestice. (Higgs: $\frac{-i}{p^2-m^2}$)

Anihilacija $e^-e^+ \to \mu^-\mu^+$ u QED



$$p_1+p_2=p_3+p_4=q$$

$$i\mathcal{M} = \bar{v}(p_2)\left(-i\underbrace{Q_e}_{-1} e\gamma^\mu\right)u(p_1)\,\frac{-ig_{\mu\nu}}{q^2}\,\bar{u}(p_3)\left(-i\underbrace{Q_\mu}_{-1} e\gamma^\nu\right)v(p_4)$$

$$\mathcal{M} = -\frac{e^2}{q^2}\left[\bar{v}(p_2)\gamma^\mu u(p_1)\right]\left[\bar{u}(p_3)\gamma_\mu v(p_4)\right]$$

Za račun σ: $|\mathcal{M}|^2 = \underbrace{\mathcal{M}^*}_{\mathcal{M}^+}\mathcal{M}$

$$(\bar{\psi}\Gamma\phi)^+ = (\psi^+\gamma^0\Gamma\phi)^+ = \underbrace{\phi^+}_{\gamma^0\gamma^0}\Gamma^+\underbrace{\gamma^{0+}}_{\gamma^0}\psi = \bar{\phi}\underbrace{\gamma^0\Gamma^+\gamma^0}_{\bar{\Gamma}}\psi \overset{SM,QED}{=} \bar{\psi}\Gamma\phi$$

spinor, 4x4 matrica

$\Gamma = \gamma^\mu$

$\gamma^{\mu +} = \gamma^0\gamma^\mu\gamma^0$

$\bar{\Gamma} = \Gamma = \gamma^\mu$

$$\mathcal{M}^+ = -\frac{e^2}{q^2}\left[\bar{u}(p_1)\gamma^\nu v(p_2)\right]\left[\bar{v}(p_4)\gamma_\nu u(p_3)\right]$$

$$|\mathcal{M}|^2 \longrightarrow \frac{1}{2}\sum_{s_1}\cdot\frac{1}{2}\sum_{s_2}\sum_{s_3}\sum_{s_4}|\mathcal{M}|^2 \equiv \overline{|\mathcal{M}|^2} \equiv \langle|\mathcal{M}|^2\rangle$$

usrednjeni kvadrat amplitude

$$\overline{|\mathcal{M}|^2} = \frac{e^4}{4q^4}\overbrace{\sum_{s_1,s_2}\left(\bar{v}(p_2,s_2)\gamma^\mu u(p_1,s_1)\right)\left(\bar{u}(p_1,s_1)\gamma^\nu v(p_2,s_2)\right)}^{\equiv L^{\mu\nu}_{(e)}} \times$$

$$\times\underbrace{\sum_{s_3,s_4}\left(\bar{u}(p_3,s_3)\gamma_\mu v(p_4,s_4)\right)\left(\bar{v}(p_4,s_4)\gamma_\nu u(p_3,s_3)\right)}_{\equiv L^{(\mu)}_{\mu\nu}}$$

„Casimirov trik"

$A,B,C,D = 1,2,3,4$

$$L^{\mu\nu}_{(e)} = \sum_{s_1,s_2}\bar{v}(p_2,s_2)_A\,\gamma^\mu_{AB}\,u(p_1,s_1)_B\,\bar{u}(p_1,s_1)_C\,\gamma^\nu_{CD}\,v(p_2,s_2)_D$$

$$= \sum_{s_1,s_2}\underbrace{v(p_2,s_2)_D\,\bar{v}(p_2,s_2)_A}_{(\not{p}_2 - m_e)_{DA}}\,\gamma^\mu_{AB}\,\underbrace{u(p_1,s_1)_B\,\bar{u}(p_1,s_1)_C}_{=(\not{p}_1 + m_e)_{BC}}\,\gamma^\nu_{CD}$$

(relacija kompletnosti)

$$= \mathrm{Tr}\,(\not{p}_2 - m_e)\gamma^\mu(\not{p}_1 + m_e)\gamma^\nu$$

$$L^{(\mu)}_{\mu\nu} = \mathrm{Tr}\,(\not{p}_3 + m_\mu)\gamma_\mu(\not{p}_4 - m_\mu)\gamma_\nu$$

$m_e \sim 0.5$ MeV $\quad m_\mu \sim 100$ MeV $\quad , E_e \gg m_e$

Approx: $\quad m_e \approx 0 \quad m_\mu \equiv M$

$$L^{\mu\nu}_{(e)} = \mathrm{Tr}\, \not{p}_2 \gamma^\mu \not{p}_1 \gamma^\nu = p_{2\rho} p_{1\sigma} \underbrace{\mathrm{Tr}\, \gamma^\rho \gamma^\mu \gamma^\sigma \gamma^\nu}_{4(g^{\rho\mu} g^{\sigma\nu} - g^{\rho\sigma} g^{\mu\nu} + g^{\rho\nu} g^{\mu\sigma})}$$

$$= 4\left(p_2^\mu p_1^\nu - (p_1 \cdot p_2) g^{\mu\nu} + p_2^\nu p_1^\mu \right)$$

$$L^{(\mu)}_{\mu\nu} = \mathrm{Tr}\, (\not{p}_3 + M) \gamma_\mu (\not{p}_4 - M) \gamma_\nu$$

$$= \underbrace{\mathrm{Tr}\, (\not{p}_3 \gamma_\mu \not{p}_4 \gamma_\nu)}_{4(p_{3\mu} p_{4\nu} - p_3 \cdot p_4 g_{\mu\nu} + p_{3\nu} p_{4\mu})} - M \underbrace{\mathrm{Tr}\, \not{p}_3 \gamma_\mu \gamma_\nu}_{=0} + M \underbrace{\mathrm{Tr}\, \gamma_\mu \not{p}_4 \gamma_\nu}_{=0} - M^2 \underbrace{\mathrm{Tr}\, \gamma_\mu \gamma_\nu}_{4 g_{\mu\nu}}$$

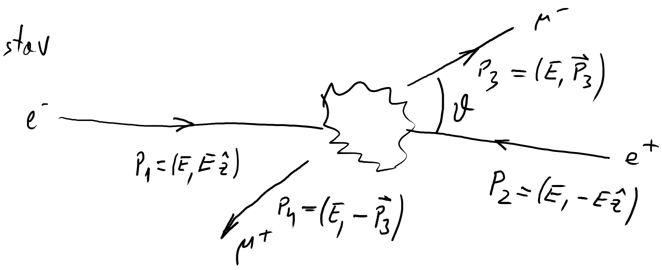
$$= 4\left[p_{3\mu} p_{4\nu} + p_{3\nu} p_{4\mu} - g_{\mu\nu} p_3 \cdot p_4 - g_{\mu\nu} M^2 \right] \qquad g_{\mu\nu} g^{\mu\nu} = 4$$

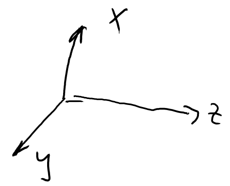
$$\overline{|\mathcal{M}|^2} = \frac{e^4}{4 q^4} \cdot 16 \Big(2(p_1 \cdot p_4)(p_2 \cdot p_3) + 2(p_1 \cdot p_3)(p_2 \cdot p_4) - \cancel{2(p_1 \cdot p_2)(p_3 \cdot p_4)} + \cancel{4(p_1 \cdot p_2)(p_3 \cdot p_4)} - \cancel{2(p_1 \cdot p_2)(p_3 \cdot p_4)}$$

$$- \underbrace{M^2 \big(2 p_2 \cdot p_1 - 4 (p_1 \cdot p_2) \big)}_{+2M^2 (p_1 \cdot p_2)} \Big)$$

$$\overline{|\mathcal{M}|^2} = \frac{8 e^4}{q^4} \Big((p_1 \cdot p_4)(p_2 \cdot p_3) + (p_1 \cdot p_3)(p_2 \cdot p_4) + M^2 (p_1 \cdot p_2) \Big) \qquad q = p_1 + p_2$$

CM sustav





$$|\vec{p}_3| = \sqrt{E^2 - M^2} \equiv \beta E, \quad \beta = \frac{v}{c}$$

$p_1 = (E, 0, 0, E)$
$p_2 = (E, 0, 0, -E)$
$p_3 = (E, \beta E \sin\vartheta, 0, \beta E \cos\vartheta)$
$p_4 = (E, -\beta E \sin\vartheta, 0, -\beta E \cos\vartheta)$

$$(p_1 \cdot p_3) = E^2 - E^2 \beta \cos\vartheta = E^2 (1 - \beta \cos\vartheta) = p_2 \cdot p_4$$

$$p_1 \cdot p_4 = E^2 (1 + \beta \cos\vartheta) = p_2 \cdot p_3$$

$$p_1 \cdot p_2 = E^2 - (-E)E = 2E^2 \quad , \quad q^2 = (p_1 + p_2)^2 = \underbrace{p_1^2}_{0} + 2 p_1 \cdot p_2 + \underbrace{p_2^2}_{0} = s = 4E^2$$

$$\overline{|\mathcal{M}|^2} = \frac{8 e^4}{(4E^2)^2} \Big(\underbrace{E^4 (1 + \beta \cos\vartheta)^2 + E^4 (1 - \beta \cos\vartheta)^2}_{2E^4 + 2E^4 \beta^2 \cos^2\vartheta} + \underbrace{M^2 \cdot 2E^2}_{\underbrace{E^2 - \vec{p}^2 = E^2 - \beta^2 E^2}_{2E^4 (1 - \beta^2)}} \Big) = e^4 \left(2 + \beta^2 \cos^2\vartheta - \beta^2 \right)$$

$$\beta = \frac{|\vec{p}|}{E} = \frac{\sqrt{E^2 - M^2}}{E} = \sqrt{1 - \frac{M^2}{E^2}} = \sqrt{1 - \frac{4M^2}{s}}$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\,\frac{|\vec{p}_f|}{|\vec{p}_i|}\,\overline{|\mathcal{M}|^2} \qquad |\vec{p}_i| = E \quad (m_e \to 0)$$
$$|\vec{p}_f| = |\vec{p}_3| = \beta E$$

$$= \frac{1}{64\pi^2 s}\,\frac{\beta \cancel{E}}{\cancel{E}} \cdot e^4 \left(2 + \beta^2\cos^2\vartheta - \beta^2\right)$$

$$\frac{e^2}{4\pi} \equiv \alpha \sim \frac{1}{137.0}$$
$$\frac{e^4}{16\pi^2} = \alpha^2$$

$$= \frac{\alpha^2}{4s}\,\beta\left(2 + \beta^2\cos^2\vartheta - \beta^2\right)$$

$$d\Omega = d(\cos\vartheta)\cdot \underbrace{d\varphi}_{2\pi}$$

$$\int_{-1}^{1} d(\cos\vartheta)\cdot 1 = 2$$
$$\int_{-1}^{1} d\cos\vartheta \cdot \cos^2\vartheta = \int_{-1}^{1} dx \cdot x^2 = \frac{x^3}{3}\Big|_{-1}^{1} = \frac{1}{3} - \frac{(-1)}{3} = \frac{2}{3}$$

$$\underline{\sigma} = \frac{\alpha^2}{4s}\,\beta \cdot 2\pi\,\beta\left(2\cdot 2 + \underbrace{\beta^2\cdot\frac{2}{3} - \beta^2\cdot 2}_{-\frac{4}{3}\beta^2}\right) = \underline{\underline{\frac{2\pi\alpha^2}{3s}\,\beta\left(3-\beta^2\right)}}$$

$$\beta \approx 1 \quad \sigma = \frac{4\pi\alpha^2}{3s}$$
$$\beta \ll 1 \quad \sigma = \frac{2\pi\alpha^2}{s}\beta \qquad [\sigma] = [\mathrm{cm}^2] = \frac{1}{\mathrm{GeV}^2} \qquad s = (p_1+p_2)^2 = 4E^2$$

$m, M \to 0$

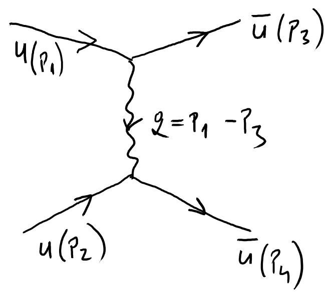
$$\overline{|\mathcal{M}|^2} = \frac{8e^4}{q^4}\left((p_1\cdot p_3)(p_2\cdot p_4) + (p_1\cdot p_4)(p_2\cdot p_3)\right)$$

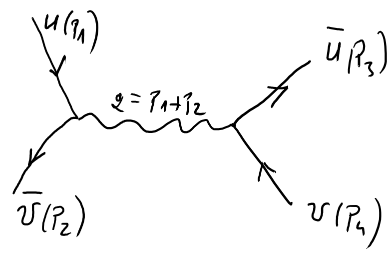
$$s = (p_1+p_2)^2 = 2p_1\cdot p_2 = 2p_3\cdot p_4 = q^2$$
$$t = (p_1-p_3)^2 = -2p_1\cdot p_3 = -2p_2\cdot p_4$$
$$\underline{u = -2p_1\cdot p_4 = -2p_2\cdot p_3}$$

$$\overline{|\mathcal{M}|^2} = \frac{8e^4}{s^2}\left(\left(-\frac{t}{2}\right)^2 + \left(-\frac{u}{2}\right)^2\right) = 2e^4\,\frac{t^2+u^2}{s^2}$$

Križna (crossing) simetrija

Promotrimo $e^-\mu^- \to e^-\mu^-$ usporedba: $e^-e^+ \to \mu^-\mu^+$





amplitudu možemo dobiti zamjenom

$$\left.\begin{array}{l}
\text{u tragu}\left\{\begin{array}{l}
p_1 \to p_1 \\
p_2 \to p_3 \ \& \ -m \text{ u } +m \iff p_2 \to -p_3 \\
p_3 \to p_4 \\
p_4 \to p_2 \ \& \ -M \text{ u } M \iff p_4 \to -p_2
\end{array}\right. \\
\text{u prop.} \quad p_2 \to -p_3
\end{array}\right\}$$

u amplitudi:

$p_1 \to p_1$

$p_2 \to -p_3$

$p_3 \to p_4$

$p_4 \to -p_2$

$s \to t, \ t \to u, \ u \to s$

Križna simetrija: amplituda se ne mijenja ako česticu iz konačnog (početnog) stanja prebacimo u antičesticu u početnom (konačnom) stanju uz zamjenu predznaka 4-impulsa

$$\mathcal{M}\left(e^-(p_1)\,\mu^-(p_2) \to e^-(p_3)\,\mu^-(p_4)\right) = \mathcal{M}\left(e^-(p_1)\,e^+(-p_3) \to \mu^-(p_4)\,\mu^+(-p_2)\right)$$

$$|\mathcal{M}(e^-\mu^- \to e^-\mu^-)|^2 = \left\{\begin{array}{c} \text{zamjena} \\ s\to t,\ t\to u,\ u\to s \\ \text{u} \\ 2e^4\,\dfrac{t^2+u^2}{s^2} \end{array}\right\} = 2e^4\,\frac{u^2+s^2}{t^2}$$

Raspršenje $1+2 \to 3+4$ u sustavu mirne mete

$\vec{p}_2 = 0 \qquad E_2 = m_2 \qquad$ (tzv. „laboratorijski sustav")

U CM sustavu smo pokazali

$$\frac{d\sigma}{d\Omega_{CM}} = \frac{1}{64\pi^2 s} \frac{|\vec{p}_f^{\,CM}|}{|\vec{p}_i^{\,CM}|} \overline{|\mathcal{M}_{fi}|^2}$$

Napišimo ovo pomoću Lorentz-invarijantnih veličina

$$t = (p_1 - p_3)^2 = p_1^2 + p_3^2 - 2p_1 \cdot p_3 = \{\text{u CM sustavu}\}$$

$$= m_1^2 + m_3^2 - 2E_1^{CM} E_3^{CM} + 2\underbrace{|\vec{p}_1^{\,CM}|}_{\substack{\parallel \\ \vec{p}_i^{\,CM}}} \underbrace{|\vec{p}_3^{\,CM}|}_{\substack{\parallel \\ \vec{p}_f^{\,CM}}} \cos\vartheta_{CM}$$

$$dt = 2|\vec{p}_i^{\,CM}||\vec{p}_f^{\,CM}|\, d\cos\vartheta_{CM}$$

(u CM sustavu $|\vec{p}_f^{\,CM}|$ ne ovisi o ϑ_{CM}. To nije slučaj u sustavu mirne mete kao što će biti važno kasnije.)

$$d\sigma = \frac{1}{64\pi^{\cancel{2}} s} \frac{\cancel{|\vec{p}_f^{\,CM}|}}{|\vec{p}_i^{\,CM}|} \overline{|\mathcal{M}_{fi}|^2} \frac{dt}{\cancel{2}\cancel{|\vec{p}_f^{\,CM}|}|\vec{p}_i^{\,CM}|} \underbrace{d\varphi}_{=\cancel{2\pi}}$$ (jer ništa ne ovisi o φ)

$$= \frac{1}{64\pi s |\vec{p}_i^{\,CM}|^2} \overline{|\mathcal{M}_{fi}|^2}$$

Za raspade smo imali $\quad |\vec{p}_i^{\,CM}| = \frac{1}{2m_a}\underbrace{\sqrt{[m_a^2 - (m_1+m_2)^2][m_a^2 - (m_1-m_2)^2]}}_{\equiv \lambda^{1/2}(m_a^2, m_1^2, m_2^2)}$

(„kinematička funkcija")

Ovdje: $m_a \to \sqrt{s}$ tj. $|\vec{p}_i^{\,CM}| = \frac{\lambda^{1/2}(s, m_1^2, m_2^2)}{2\sqrt{s}}$

$$\Rightarrow \boxed{\frac{d\sigma}{dt} = \frac{\overline{|\mathcal{M}_{fi}|^2}}{16\pi\, \lambda(s, m_1^2, m_2^2)}}$$

(vrijedi u svim sustavima)

Raspršenje $e^-\mu^- \to e^-\mu^-$ (u sustavu mirne mete)

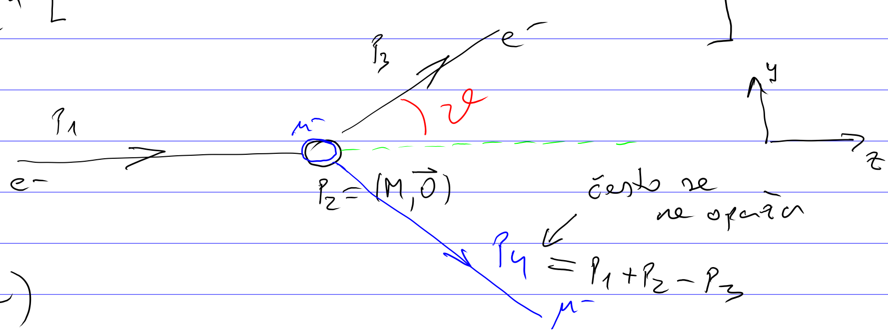
Za $e^-e^+ \to \mu^-\mu^+$ smo imali (uz $m_e = 0$, $m_\mu = M$ što i ovdje pretpostavljamo)

$$\overline{|\mathcal{M}(e^-e^+\to\dots)|^2} = \frac{8e^4}{q^4}\left[(p_1\cdot p_4)(p_2\cdot p_3) + (p_1\cdot p_3)(p_2\cdot p_4) + M^2(p_1\cdot p_2)\right]$$

Križna simetrija daje onda za $e^-\mu^- \to e^-\mu^-$

$$\overline{|\mathcal{M}|^2} = \left\{\begin{array}{l} p_1 \to p_1 \\ p_2 \to -p_3 \\ p_3 \to p_4 \\ p_4 \to -p_2 \end{array}\right\} = \frac{8e^4}{q^4}\left[(p_1\cdot p_2)(p_3\cdot p_4) + (p_1\cdot p_4)(p_2\cdot p_3) - M^2(p_1\cdot p_3)\right]$$

$$\boxed{q = p_1 - p_3}$$



$p_1 = (E_1, 0, 0, E_1)$

$p_3 = (E_3, 0, E_3\sin\vartheta, E_3\cos\vartheta)$

$p_1\cdot p_2 = E_1 M$

$p_2\cdot p_3 = E_3 M$

$p_1\cdot p_3 = E_1E_3(1-\cos\vartheta)$

$p_3\cdot p_4 = p_3\cdot p_1 + p_3\cdot p_2 - \underbrace{p_3^2}_{m_e^2=0} = E_1E_3(1-\cos\vartheta) + E_1 M$

$p_1\cdot p_4 = \underbrace{p_1\cdot p_1}_{0} + p_1\cdot p_2 - p_1\cdot p_3 = E_1M - E_1E_3(1-\cos\vartheta)$

$$\overline{|\mathcal{M}|^2} = \frac{8e^4}{q^2}\left[E_1M\left(E_1E_3(1-\cos\vartheta) + E_3M\right) + E_3M\left(E_1M - E_1E_3(1-\cos\vartheta)\right) - M^2E_1E_3(1-\cos\vartheta)\right]$$

$$= \frac{8e^4}{q^2} ME_1E_3\left[\underbrace{E_1(1-\cos\vartheta) + M - E_3(1-\cos\vartheta) + M - M(1-\cos\vartheta)}_{(E_1-E_3)\underbrace{(1-\cos\vartheta)}_{2\sin^2\frac{\vartheta}{2}} + M\underbrace{(1+\cos\vartheta)}_{2\cos^2\frac{\vartheta}{2}}}\right]$$

$$q^2 = (p_1-p_3)^2 = \underbrace{2m_e^2}_{=0} - 2p_1\cdot p_3 = -2E_1E_3(1-\cos\vartheta) = -4E_1E_3\sin^2\frac{\vartheta}{2} < 0$$

$\boxed{Q^2 \equiv -q^2}$ prijenos impulsa

$\boxed{E_1 - E_3 \equiv \nu}$ gubitak energije elektrona > 0 (u ovom sustavu)

Ove dvije varijable nisu nezavisne :

$$q \cdot P_2 = (P_1 - P_3)\cdot P_2 = E_1 M - E_3 M = \nu M$$

$$(q+P_2)^2 = (P_1 - P_3 + P_2)^2 = P_4^2 = M^2$$

$$= \underbrace{q^2}_{-Q^2} + \underbrace{P_2^2}_{M^2} + 2q\cdot P_2$$

$$Q^2 = 2q\cdot P_2$$

$$\boxed{\nu = \frac{Q^2}{2M}}$$

$$\overline{|M|^2} = \underbrace{\frac{\cancel{8}e^4}{(-\cancel{4}E_1E_3\sin^2\vartheta/_2)^2} M E_1 E_3 \cdot \cancel{2}M}_{\frac{e^4M^2}{E_1E_3\sin^4\frac{\vartheta}{2}}} \left[\cos^2\frac{\vartheta}{2} + \frac{Q^2}{2M^2}\sin^2\frac{\vartheta}{2}\right]$$

$$\frac{d\sigma}{dt} = \frac{\overline{|M_{fi}|^2}}{16\pi\,\lambda(s,m_1^2,m_2^2)} \quad ; \quad \lambda(s,m_1^2=0,m_2^2=M^2) = (s-M^2)^2 = (2ME_1)^2 = 4M^2E_1^2$$

$$\left\{ s = (P_1+P_2)^2 = m_e^2 + M^2 + 2P_1\cdot P_2 = M^2 + 2ME_1 \right\}$$

No obično se mjeri $\frac{d\sigma}{d\Omega} = \frac{d\sigma}{dt}\cdot\frac{dt}{d\Omega} = \frac{d\sigma}{dt}\cdot\frac{1}{2\pi}\frac{dt}{d(\cos\vartheta)}$

$$t = (P_1-P_3)^2 = q^2 = -Q^2 = -2M\nu = -2M(E_1-E_3)$$

$$= -2E_1E_3(1-\cos\vartheta)$$

$\{ E_1$ fiksno, $E_3 = E_3(\vartheta)$

$$-\cancel{2}ME_1 + \cancel{2}ME_3 + \cancel{2}E_1E_3(1-\cos\vartheta) = 0$$

$$E_3 = \frac{ME_1}{M + E_1(1-\cos\vartheta)}$$

$$\frac{dt}{d\cos\vartheta} = 2M\frac{dE_3}{d\cos\vartheta} = 2M\frac{(-1)(-E_1)ME_1}{[M+E_1(1-\cos\vartheta)]^2} = 2E_3^2$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{2\pi}2E_3^{\cancel{2}}\frac{1}{16\pi\,4\cancel{M^2}E_1^2}\frac{e^4\cancel{M^2}}{E_1\cancel{E_3}\sin^4\frac{\vartheta}{2}}\left(\cos^2\frac{\vartheta}{2} + \frac{Q^2}{2M^2}\sin^2\frac{\vartheta}{2}\right)$$

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2\sin^4\frac{\vartheta}{2}}\frac{E_3}{E_1}\left(\cos^2\frac{\vartheta}{2} + \frac{Q^2}{2M^2}\sin^2\frac{\vartheta}{2}\right)}$$

ϑ je jedina „slobodna" varijabla koja se mjeri

$Q^2 = 2M\nu = 2M(E_1-E_3)$

$E_3 = E_3(\vartheta)$

(mjerenje E_3 omogućuje kontrolu da meta nije razbijena tj. da je $P_4^2 = M^2$ što je važno za $ep \to ep$ raspršenje.)

Granični slučajevi

1. $m_e \ll E_1 \ll M$: e^- je relativističk, zanemarujemo odboj mete (prijenos impulsa na metu)

$$E_3 = \frac{M E_1}{M + \underbrace{E_1(1-\cos\vartheta)}_{\ll M}} \approx E_1 \quad ; \quad Q^2 = 2M(E_1 - E_3) \ll M^2$$

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2 \sin^4\frac{\vartheta}{2}} \cos^2\frac{\vartheta}{2}}$$ Mottovo raspršenje

2. $E_1 \approx m_e \ll M$: e^- nerelativističk.

Da nismo zanemarivali m_e dobili bi

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4|\vec{p}_1|^2 \sin^4\frac{\vartheta}{2}} \left(\frac{m_e^2}{|\vec{p}_1|^2} + \cos^2\frac{\vartheta}{2} \right)$$

(Hiperrelativistički $|\vec{p}_1| = E_1 \gg m_e$ reproducira Mottovu formulu)

$$\frac{\vec{p}_1^{\,2}}{2m_e} = E_K \qquad \boxed{\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{16 E_K^2 \sin^4\frac{\vartheta}{2}}}$$ Rutherfordovo raspršenje

↓

(ista kao u klasičnoj mehanici)

Potpuna formula obzirom na Mottovu uključuje

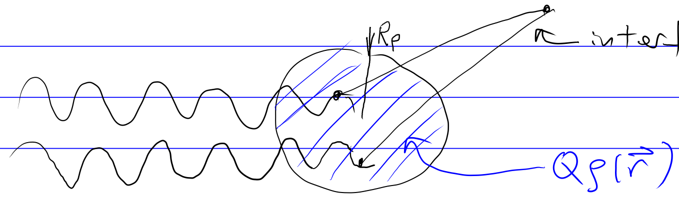
*) faktor $\frac{E_3}{E_1}$ – efekt odboja mete („recoil")

*) član $\propto \frac{Q^2}{2M^2}$ – interakcija s magnetskim momentom mete

Raspršenje elektrona na protonu

Pri malim E_1 (velika Comptonova valna duljina λ) $\rightarrow$ mala rezolucija i sve je isto kao za raspršenje na μ^- osim što je $M \sim 1\,\mathrm{GeV}$ (predznak naboja mete je nebitan)

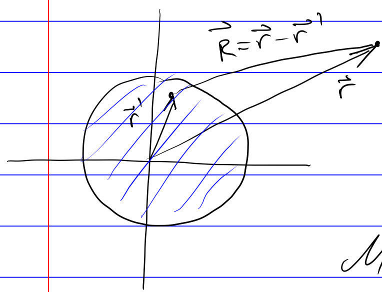
Za dovoljno velike E_1 (reda veličine $\frac{1}{R_p}$, bit ćemo precizniji kasnije) λ je mala, projektil počinje razlučivati metu i ona se više ne može smatrati točkastom.



$$\int \rho(\vec{r})\, d^3 r = 1$$

← raspodjela naboja

Raspršenje na općenitoj raspodjeli naboja



$$V(\vec{r}) = \int \frac{Q\rho(\vec{r}\,')}{4\pi|\vec{R}|}\, d^3 r'$$

$$\mathcal{M}_{fi} = \langle f | V | i \rangle \propto \int d^3 r\, e^{-i\vec{p}_3\cdot\vec{r}}\, \frac{Q\rho(\vec{r}\,')\, d^3 r'}{4\pi|\vec{R}|}\, e^{+i\vec{p}_1\cdot\vec{r}}$$

$$= \int d^3 r\, d^3 r'\, e^{i\vec{q}\cdot\underbrace{(\vec{r}-\vec{r}\,')}_{=\vec{R}}}\, e^{+i\vec{q}\cdot\vec{r}\,'}\, \frac{Q\rho(\vec{r}\,')}{4\pi|\vec{R}|} = \left\{ d^3 R = d^3 r \right\}$$

$$= \underbrace{\int d^3 R\, e^{i\vec{q}\cdot\vec{R}}\, \frac{Q}{4\pi|\vec{R}|}}_{\text{daje } \mathcal{M}_{fi}^{\text{točkasta meta}}} \cdot \underbrace{\int d^3 r'\, e^{i\vec{q}\cdot\vec{r}\,'}\, \rho(\vec{r}\,')}_{\equiv F(\vec{q})}$$

$\frac{Q}{4\pi|\vec{R}|}$: $V(\vec{r})$ za točkasti naboj Q

Fourierov transformat od $\rho(\vec{r})$

$\equiv F(\vec{q})$ – funkcija strukture („form faktor")

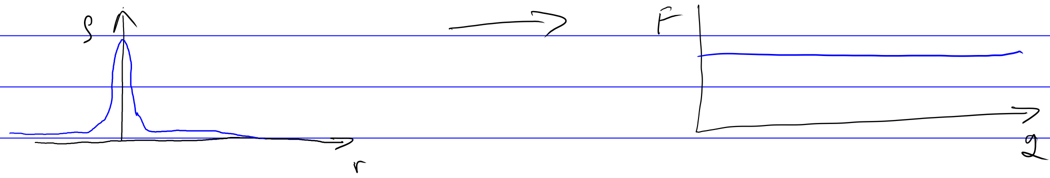
(sferno-simetrična raspodjela ili nepoznata orijentacija)

$F(\vec{q})$ je skalar, ovisi samo o $\vec{q}$ $\Rightarrow$ ovisnost može biti samo kroz ovisnost o $\vec{q}^{\,2}$ $\Rightarrow$ $F(\vec{q}) \rightarrow F(\vec{q}^{\,2})$

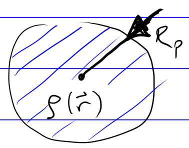
$$\boxed{\left.\frac{d\sigma}{d\Omega}\right|^{\text{Mott}}_{\text{na raspodjeli } \rho(\vec{r})} = \left.\frac{d\sigma}{d\Omega}\right|^{\text{Mott}}_{\text{na točkastoj meti}} \cdot \left|F(\vec{q}^{\,2})\right|^2}$$

*) $F(0) = \int d^3r\, e^0 \rho(\vec{r}) = 1$ (normalizacija)

*) $\rho(\vec{r}) = \delta^3(\vec{r})$ $\qquad\qquad\qquad F(\vec{q}^{\,2}) = 1$



*) Ako je $|\vec{q}| \ll \frac{1}{R_p} \sim \frac{0.2\text{ GeV fm}}{1\text{ fm}} \sim 0.2\text{ GeV}$



za proton $R_p \sim 1$ fm $\quad \Rightarrow \vec{q}\cdot\vec{r} \ll 1$ svugdje gdje je $\rho(\vec{r}) \neq 0$

onda imamo $\quad F(\vec{q}) = \int d^3r\, \underbrace{e^{i\vec{q}\cdot\vec{r}}}_{\approx 0} \rho(\vec{r}) \approx 1 \quad$ (ne „vidimo" strukturu)

*) Suprotna granica: $\vec{q} \gg \frac{1}{R_p} \Rightarrow \int d^3r\, \underbrace{e^{i\vec{q}\cdot\vec{r}}}_{\text{faza brzo varira}} \rho(\vec{r}) \to 0$

$$\Rightarrow F(\vec{q}^{\,2}) \xrightarrow[\vec{q}^{\,2}\to\infty]{} 0$$

*) Neka je raspodjela sferno simetrična $\quad \rho(\vec{r}) = \rho(r)$

$$F(\vec{q}^{\,2}) = \int d^3r\, e^{i\vec{q}\cdot\vec{r}} \rho(r) = \int r^2 dr \underbrace{d\varphi}_{2\pi}\, \underbrace{d\cos\vartheta\, e^{iqr\cos\vartheta}}_{\frac{1}{iqr}\underbrace{e^{iqr}-e^{-iqr}}_{2i\sin qr}} \rho(r) = \frac{4\pi}{q}\int_0^\infty r\,dr\, \sin qr\, \rho(r)$$

(za mali $\vec{q}^{\,2}$)

$$= \frac{4\pi}{q}\int r\,dr\left(qr - \frac{q^3r^3}{6} + \dots\right)\rho(r) = \underbrace{\int 4\pi r^2 dr\, \rho(r)}_{1} - \frac{q^2}{6}\underbrace{\int 4\pi r^2 dr \cdot r^2 \rho(r)}_{\langle r^2\rangle}$$

$$\left.\frac{dF(\vec{q}^{\,2})}{d\vec{q}^{\,2}}\right|_{\vec{q}^{\,2}=0} = 0 - \frac{1}{6}\langle r^2\rangle + 0 \quad \Rightarrow \quad \boxed{\langle r^2\rangle = -6 \left.\frac{dF(\vec{q}^{\,2})}{d\vec{q}^{\,2}}\right|_{\vec{q}^{\,2}=0}}$$

srednji kvadrat radijusa naboja

(To je ono što se smatra „radijusom" čestice.)

Za proizvoljni elastični ep sudar onda umjesto

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2 \sin^4\frac{\vartheta}{2}} \frac{E_3}{E_1}\left(\cos^2\frac{\vartheta}{2} + \frac{Q^2}{2M^2}\sin^2\frac{\vartheta}{2}\right)$$

imamo

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2 \sin^4\frac{\vartheta}{2}} \frac{E_3}{E_1}\left(\frac{G_E^2+\tau G_M^2}{1+\tau}\cos^2\frac{\vartheta}{2} + 2\tau G_M^2 \sin^2\frac{\vartheta}{2}\right) \qquad \tau = Q^2/4M^2}$$

(Rosenbluthova formula)

$G_E(Q^2)$, $G_M(Q^2)$ – el. i magn. funkcije strukture („Sachs form factors")

$\searrow$ Lorentz invarijantne!

Za $Q^2 \ll 4M^2$

$G_E(Q^2)$ je Fourierov transformat od $\rho(\vec{r})$

$G_M(Q^2)$ —ıı— od $\mu(\vec{r})$ ← raspodjela magn. momenta

i u tom režimu G_E dominira. Za $Q^2 \gg 4M^2$, $\tau \gg 1$ i G_M dominira.

Za točkaste Diracove čestice $G_E(Q^2) = G_M(Q^2) = 1$ i iz Rosenbluthove dobivamo prijašnju formulu.

Za proton: $G_E(0) = 1$ (ukupni naboj)

$G_M(0) = 2.79$ (ukupni magn. moment protona je „anomalan" jer se razlikuje od magn. momenta elementarne Diracove čestice. To ukazuje na podstrukturu.)

Eksperimenti na protonu:

$$G_M(Q^2) \approx 2.79\, G_E(Q^2) \approx 2.79 \frac{1}{\left(1+\frac{Q^2}{0.71\,\mathrm{GeV}^2}\right)^2} \qquad \text{(dipolna formula)}$$

$$\langle r^2 \rangle = -6 \cdot \left.\frac{dG_E(Q^2)}{dQ^2}\right|_{Q^2=0} = -6 \cdot (-2) \cdot \frac{1}{0.71\,\mathrm{GeV}^2} \cdot (0.2\,\mathrm{GeV\,fm})^2 = 0.68\,\mathrm{fm}^2$$

$\longrightarrow \sqrt{\langle r^2 \rangle} = 0.8\,\mathrm{fm}$ („radijus protona")

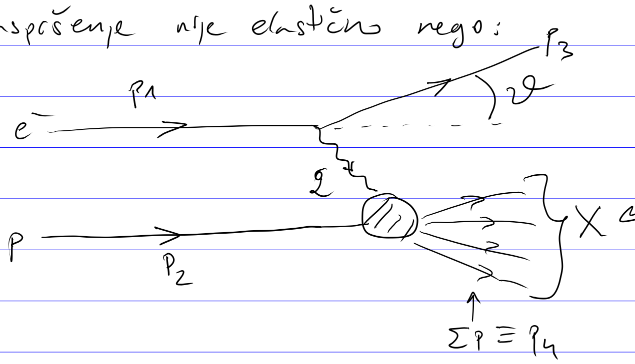
Duboko neelastično raspršenje (DIS)

Elastično ep raspršenje:

Rosenbluthova formula $$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2 \sin^4\frac{\vartheta}{2}}\,\frac{E_3}{E_1}\left(\frac{G_E^2+\tau G_M^2}{1+\tau}\cos^2\frac{\vartheta}{2} + 2\tau G_M^2 \sin^2\frac{\vartheta}{2}\right)$$

Osim o E_1 (energija snopa) $d\sigma$ ovisi o samo jednoj kinematičkoj varijabli: Q^2 (Možemo uzeti bilo koju iz skupa $\{Q^2, E_3, \vartheta\}$; ostale dvije su dane preko te izabrane, zahvaljujući zakonima očuvanja $p_1+p_2=p_3+p_4$.)

Ako raspršenje nije elastično nego:



može biti:
- nekoliko čestica (razbijeni proton)
- jedna čestica (Δ, ...) (pobuđeno stanje protona) $\Delta \to p+\pi$

$p_4^2 \equiv W^2 > m_p^2$ je druga nezavisna varijabla o kojoj ovisi σ.

↑ (Zakon očuvanja barionskog broja + činjenica da je proton najlakši barion.)

No umjesto W^2 koristit ćemo

$$\boxed{x = \frac{Q^2}{2p_2\cdot q}}$$ „Bjorkenov x"

$$W^2 = (p_2+q)^2 = \underbrace{p_2^2}_{m_p^2} + 2p\cdot q + \underbrace{q^2}_{-Q^2} \Rightarrow x = \frac{Q^2}{W^2+Q^2-m_p^2}$$

$W^2 > m_p^2 \Rightarrow x<1$ ($x=1$ za elastično ep raspršenje)

$$\boxed{0 \leq x \leq 1}$$

$$\frac{d\sigma}{dQ^2dx} = \frac{4\pi\alpha^2}{Q^4}\left[(1-y)\frac{F_2(x,Q^2)}{x} + y^2 F_1(x,Q^2)\right]$$

ovisnost o dvije varijable

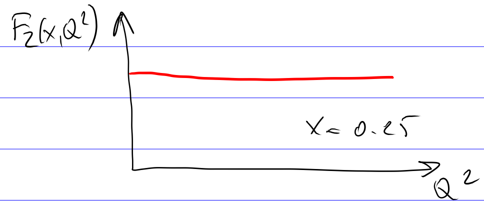
E+M

M

$$Q^2 = xy(s - m_p^2)$$

Ovo je analogno Rosenbluthovoj formuli, ali $F_{1,2}(x,Q^2)$ ovise o 2 varijable i nisu F.T. neke distribucije $\rho(\vec{x})$.

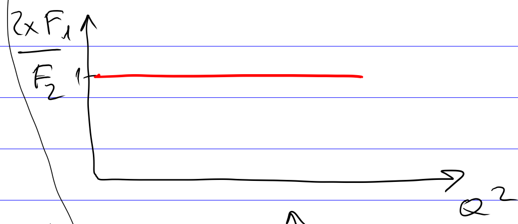
DIS eksperimenti (SLAC 1968):



„Bjorkenovo skaliranje" (nestaje ovisnost o Q^2)

Posljedica raspršenja na točkastim konstituentima protona („partoni").
(Slično kao kod $G_E(Q^2) \to$ const za $Q^2 \ll m_p^2$ pa cijeli proton izgleda točkasto.)

*



Callan-Gross relacija
$2xF_1 = F_2$
(posljedica toga što su partoni spina 1/2)

Što je x u režimu Bjorkenovog skaliranja?

U sustavu u kojem je $E_2 \gg m_p$ i zanemarujemo transverzalne impulse:

$$\cancel{m_q^2} = (q+\xi p_2)^2 = \underbrace{q^2}_{-Q^2} + 2\xi q\cdot p_2 + \underbrace{(\xi p_2)^2}_{\cancel{m_p^2}} \quad \Rightarrow \quad \xi = \frac{Q^2}{2p_2\cdot q} = x$$

→ Bjorkenov x je frakcija impulsa protona koju ima parton na kojem se dogodi raspršenje.

FEČ 2:
Gell-Mann je ranije objasnio spektar hadrona postulirajući kvarkove.
Jaka interakcija kvarkova ide izmjenom „gluona".

partoni = kvarkovi i gluoni

Raspršenje na kvarku naboja Q_q je analogno $e\mu^-$ raspršenju

$$\frac{d\sigma}{dQ^2} = \frac{4\pi\alpha^2}{Q^4} Q_q^2 \left[(1-y) + \frac{y^2}{2}\right]$$

(Dobije se iz formule koju znamo zamjenom varijabli.)

Ako je $q(x)dx$ broj kvarkova q s impulsom između x i $x+dx$ nekoherentni zbroj svih doprinosa je:

$$\frac{d\sigma}{dQ^2} = \frac{4\pi\alpha^2}{Q^4} \sum_{i=u,\bar{u},d,\bar{d},\ldots} q_i(x)\, Q_i^2 \left[(1-y) + \frac{y^2}{2}\right] dx$$

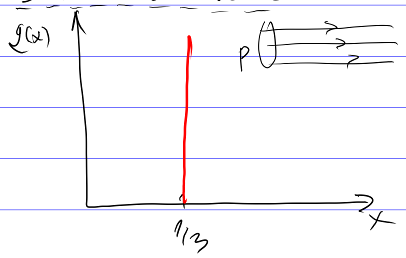
$$\Rightarrow \quad F_2(x,Q^2) = 2x F_1(x,Q^2) = x \sum_i Q_i^2\, q_i(x)$$

Mjerimo u DIS
e, νp, en, ...

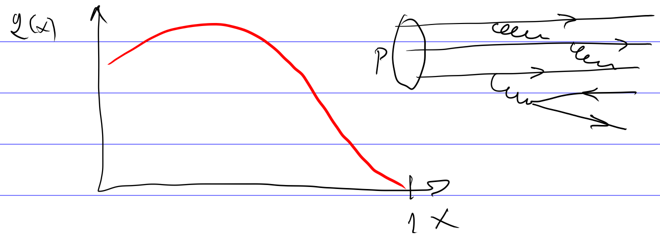
„partonske distribucijske funkcije" (PDF)

fitanje na mjerene $F_{1,2}$

3 slobodna kvarka



interagirajući kvarkovi



$u(x) = u_v(x) + u_s(x)$ valentni + „kvarkovi mora"

$\bar{u}(x) = u_s(x)$

$$\int_0^1 dx\, u_v(x) = 2 \qquad \int_0^1 dx\, d_v(x) = 1$$

(sumacijska pravila)

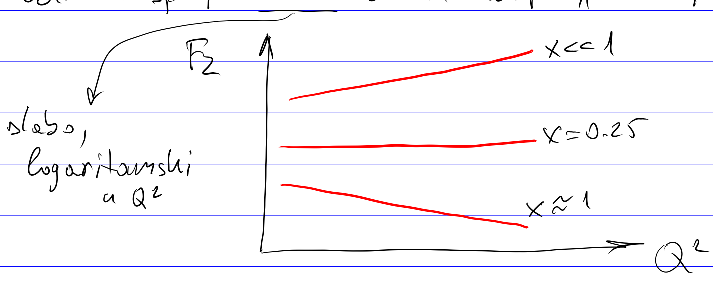
U tom smislu proton ima dva u i jedan d kvark.

Eksperiment:

$$\int x\,(u(x)+\bar{u}(x))\,dx = 0.36 \ ; \quad \int x\,(d(x)+\bar{d}(x))\,dx = 0.18$$

$\approx 50\%$ impulsa otpada na gluone.

Oblik raspodjele ovisi o Q^2. (Bolja „rezolucija" → vidimo više partona.)



$F_2(x),\ q(x) \longrightarrow F_2(x,Q^2),\ q(x,Q^2)$

odstupanje od
Bjorkenovog skaliranja