Razne formule

« Hidrodinamika »

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Pregled predavanja

Posebne matrice

Prostorne derivacije

Gradijent u raznim koordinatnim sustavima

Divergencija u raznim koordinatnim sustavima

Rotacija u raznim koordinatnim sustavima

Laplacian u raznim koordinatnim sustavima

Razne jednadžbe

Posebne matrice

1. Kronekerov simbol, jedinična matrica

$$\delta_{ij} = \begin{cases} 1 & \text{za } i = j \\ 0 & \text{za } i \neq j \end{cases} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2. Antisimetrična matrica

$$e_{ijk} = \begin{cases} +1 & \text{za } i,j,k = 1,2,3 \quad \text{ciklička permutacija} \\ -1 & \text{za } i,j,k = 3,2,1 \quad \text{ciklička permutacija} \\ \ \ 0 & \text{za } i = j \text{ ili } j = k \text{ ili } k = i \end{cases}$$

Vrijedi

$$\vec{a} = \sum_i a_i \vec{e}_i, \qquad a_i = a_x, a_y, a_z$$

$$\vec{b} = \sum_i b_i \vec{e}_i, \qquad b_i = b_x, b_y, b_z$$

$$\vec{c} = \vec{a} \times \vec{b}, \qquad c_i = \sum_{j,k} e_{ijk}\ a_j\ b_k$$

Prostorne derivacije

1. Gradijent

$$\vec{\nabla} n = \sum_i \vec{e_i}\,\frac{\partial n}{\partial x_i} = \vec{i}\,\frac{\partial n}{\partial x} + \vec{j}\,\frac{\partial n}{\partial y} + \vec{k}\,\frac{\partial n}{\partial z}$$

2. Divergencija

$$\vec{\nabla} \cdot \vec{v} = \sum_i \frac{\partial v_i}{\partial x_i} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

3. Rotacija

$$\vec{\nabla} \times \vec{v} = \sum_{i,j,k} e_{ijk}\;\vec{e_i}\,\frac{\partial v_j}{\partial x_k} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_x \end{vmatrix}$$

4. Laplaceov operator

$$\Delta n = \frac{\partial^2 n}{\partial x^2} + \frac{\partial^2 n}{\partial y^2} + \frac{\partial^2 n}{\partial z^2}$$

Kombinacija derivacija skalarnih i vektorskih polja

1. $\vec{\nabla} \cdot \vec{\nabla} = \Delta$
2. $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{v}) \equiv 0$
3. $\vec{\nabla} \times (\vec{\nabla} n) \equiv 0$
4. $\vec{\nabla}(n\ \vec{v}) = n\ (\vec{\nabla} \cdot \vec{v}) + (\vec{\nabla} n) \cdot \vec{v}$
5. $\vec{\nabla} \times (n\ \vec{v}) = n\ (\vec{\nabla} \times \vec{v}) + (\vec{\nabla} n) \times \vec{v}$
6. $\vec{\nabla} \times (\vec{\nabla} \times \vec{v}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{v}) - (\vec{\nabla} \cdot \vec{\nabla})\vec{v} = \vec{\nabla}(\vec{\nabla} \cdot \vec{v}) - \Delta\vec{v}$
7. $\vec{\nabla} \frac{\vec{v}^2}{2} = \vec{v} \times (\vec{\nabla} \times \vec{v}) + (\vec{v} \cdot \vec{\nabla})\vec{v}$

Gradijent u raznim koordinatnim sustavima

- Kartezijev

$$\vec{\nabla} n = \vec{i}\,\frac{\partial n}{\partial x} + \vec{j}\,\frac{\partial n}{\partial y} + \vec{k}\,\frac{\partial n}{\partial z}$$

- Cilindrični

$$\vec{\nabla} n = \vec{\rho}_0\,\frac{\partial n}{\partial \rho} + \vec{\varphi}_0\,\frac{1}{\rho}\,\frac{\partial n}{\partial \varphi} + \vec{k}\,\frac{\partial n}{\partial z}$$

- Sferni

$$\vec{\nabla} n = \vec{r}_0\,\frac{\partial n}{\partial r} + \vec{\varphi}_0\,\frac{1}{r\,\sin\vartheta}\frac{\partial n}{\partial \varphi} + \vec{\vartheta}_0\,\frac{1}{r}\,\frac{\partial n}{\partial \vartheta}$$

Divergencija u raznim koordinatnim sustavima

- Kartezijev ($\vec{a} = a_x \ \vec{i} + a_y \ \vec{j} + a_z \ \vec{k}$)

$$\vec{\nabla} \cdot \vec{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}$$

- Cilindrični ($\vec{a} = a_\rho \ \vec{\rho}_0 + a_\varphi \ \vec{\varphi}_0 + a_z \ \vec{k}$)

$$\vec{\nabla} \cdot \vec{a} = \frac{1}{\rho} \frac{\partial}{\partial \rho}(\rho \ a_\rho) + \frac{1}{\rho} \frac{\partial a_\varphi}{\partial \varphi} + \frac{\partial a_z}{\partial z}$$

- Sferni ($\vec{a} = a_r \ \vec{r}_0 + a_\varphi \ \vec{\varphi}_0 + a_\vartheta \ \vec{\vartheta}_0$)

$$\vec{\nabla} \cdot \vec{a} = \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 \ a_r) + \frac{1}{r \sin \vartheta} \frac{\partial a_\varphi}{\partial \varphi} + \frac{1}{r \sin \vartheta} \frac{\partial}{\partial \vartheta}(\sin \vartheta \ a_\vartheta)$$

Rotacija u raznim koordinatnim sustavima

- Kartezijev ($\vec{a} = a_x \ \vec{i} + a_y \ \vec{j} + a_z \ \vec{k}$)

$$\vec{\nabla} \times \vec{a} = \vec{i} \ (\partial_y a_z - \partial_z a_y) + \vec{j} \ (\partial_z a_x - \partial_x a_z) + \vec{k} \ (\partial_x a_y - \partial_y a_x)$$

- Cilindrični ($\vec{a} = a_\rho \ \vec{\rho}_0 + a_\varphi \ \vec{\varphi}_0 + a_z \ \vec{k}$)

$$\vec{\nabla} \times \vec{a} = \frac{1}{\rho} \begin{vmatrix} \vec{\rho}_0 & \rho \ \vec{\varphi}_0 & \vec{k} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial z} \\ a_\rho & \rho \ a_\varphi & a_z \end{vmatrix}$$

- Sferni ($\vec{a} = a_r \ \vec{r}_0 + a_\varphi \ \vec{\varphi}_0 + a_\vartheta \ \vec{\vartheta}_0$)

$$\vec{\nabla} \times \vec{a} = \frac{1}{r^2 \sin\vartheta} \begin{vmatrix} \vec{r}_0 & r \ \vec{\vartheta}_0 & r \ \sin\vartheta \ \vec{\varphi}_0 \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \vartheta} & \frac{\partial}{\partial \varphi} \\ a_r & r \ a_\vartheta & r \ \sin\vartheta \ a_\varphi \end{vmatrix}$$

Laplacian u raznim koordinatnim sustavima

- Kartezijev

$$\Delta\ n = \frac{\partial^2 n}{\partial\ x^2} + \frac{\partial^2 n}{\partial\ y^2} + \frac{\partial^2 n}{\partial\ z^2}$$

- Cilindrični

$$\Delta\ n = \frac{1}{\rho}\frac{\partial}{\partial\ \rho}\left(\rho\frac{\partial n}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 n}{\partial\ \varphi^2} + \frac{\partial^2 n}{\partial\ z^2}$$

- Sferni

$$\Delta\ n = \frac{1}{r}\frac{\partial}{\partial\ r}\left(r^2\frac{\partial n}{\partial r}\right) + \frac{1}{r^2 \sin^2 \vartheta}\frac{\partial^2 n}{\partial\ \varphi^2} + \frac{1}{r^2 \sin \vartheta}\frac{\partial}{\partial\ \vartheta}\left(\sin \vartheta\ \frac{\partial\ n}{\partial\ \vartheta}\right)$$

Jednadžbe

- Eulerova jednadžba

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v}(\vec{r}, t) = -\frac{1}{\rho}\vec{\nabla} p(\vec{r}, t) + \vec{g}$$

- Navier-Stokesova jednadžba

$$\rho \frac{D\vec{v}}{Dt} = \vec{f} - \vec{\nabla} p + \eta \, \Delta \vec{v} + (\zeta + \frac{\eta}{3}) \, \vec{\nabla}(\vec{\nabla} \cdot \vec{v})$$

- Jednadžba kontinuiteta

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \, \vec{v}) = 0$$

- Bernoullijeva jednadžba

$$\frac{v^2}{2} + \frac{p}{\rho} + g \, z = konst.$$