2 Integral

2.1 Neodređeni i određeni integral

Definicija. Neka je $I \subseteq \mathbb{R}$ otvoreni interval i $f\colon I \to \mathbb{R}$ funkcija. **Primitivna funkcija** funkcije f je funkcija $F\colon I \to \mathbb{R}$ takva da je

$$F'(x) = f(x), \quad \text{za sve } x \in I.$$

Primjer. (a) Za $f(x) = 0$ npr. imamo $F(x) = 7$.
(b) Za $f(x) = x^5$ npr. imamo $F(x) = \frac{x^6}{6} + \pi$.
(c) Za $f(x) = \frac{1}{1+x^2}$ npr. imamo $F(x) = \operatorname{arctg} x$.

Napomena. (a) Ako je F primitivna funkcija od f, onda je i $F + C$ primitivna funkcija od f, za sve $C \in \mathbb{R}$, jer je

$$(F(x) + C)' = F'(x) + 0 = f(x).$$

(b) Ako su F i G primitivne funkcije od f, onda je

$$(G(x) - F(x))' = G'(x) - F'(x) = f(x) - f(x) = 0, \text{ za sve } x \in I$$

pa postoji $C \in \mathbb{R}$ takav da je $G(x) - F(x) = C$, tj. $G(x) = F(x) + C$. Dakle, čim znamo jednu primitivnu funkciju F od f, onda znamo sve primitivne funkcije i one su oblika $F(x) + C$, za $C \in \mathbb{R}$

Definicija. Skup $\{F + C\colon C \in \mathbb{R}\}$ svih primitivnih funkcija od f zovemo **neodređeni integral** ili **antiderivacija** od f i taj skup označavamo s

$$\int f(x)\,dx = F(x) + C.$$

Zadatak 2.1.1. Izračunajte neodređene integrale:

$$\text{(a)} \int x^2\,dx \qquad \text{(b)} \int \frac{dx}{x} \qquad \text{(c)} \int 5^x\,dx \qquad \text{(d)} \int \frac{dx}{\sqrt[4]{x}}.$$

Rješenje. (a) $\displaystyle\int x^2\,dx = \frac{x^3}{3} + C$

(b) $\int \frac{dx}{x} = \ln|x| + C$

(c) $\int 5^x \, dx = \frac{5^x}{\ln 5} + C$

(d) $\int \frac{dx}{\sqrt[4]{x}} = \int x^{-\frac{1}{4}} \, dx = \frac{4}{3} x^{\frac{3}{4}} + C$

△

Definicija. Neka je $f\colon [a,b] \to \mathbb{R}$ ograničena funkcija. Ako je f Riemann-integrabilna, onda realni broj $\int\limits_a^b f(x)\,dx$ zovemo **određeni integral**.

Teorem 7. *Neka je $I \subseteq \mathbb{R}$ otoreni interval i $f\colon I \to \mathbb{R}$ neprekidna funkcija. Ako je $c \in I$ onda je s*

$$F\colon I \to \mathbb{R}, \quad F(x) = \int\limits_c^x f(t)\,dt, \; x \in I$$

definirana primitivna funkcija od f.

Teorem 8. *Neka je $I \subseteq \mathbb{R}$ otvoreni interval i $f\colon I \to \mathbb{R}$ neprekidna funkcija. Ako je F primitivna funkcija od f, onda za svaki segment $[a,b] \subseteq I$ vrijedi **Newton-Leibnizova formula**:*

$$\int\limits_a^b f(x)\,dx = F(b) - F(a) =: F(x)\Big|_a^b$$

Primjer.

$$\int_{-1}^{1} \frac{dx}{1+x^2} = \operatorname{arctg} x\Big|_{-1}^{1} = \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}$$

Zadatak 2.1.2. Izračunajte određene integrale:

(a) $\int\limits_0^{\frac{\pi}{2}} (4\sin^2 x - 3\cos^2 x)\,dx$ (b) $\int\limits_0^1 \frac{dx}{\operatorname{ch} x + \operatorname{sh} x}$ (c) $\int\limits_1^8 \frac{x^4 - 4x^3 + 2\sqrt[3]{x}}{\sqrt[5]{x^4}}$.

Rješenje. (a) $\int_0^{\frac{\pi}{2}} (4\sin^2 x - 3\cos^2 x)\,dx = \int_0^{\frac{\pi}{2}} (7\sin^2 x - 3)\,dx = \int_0^{\frac{\pi}{2}} \left(7 \cdot \frac{1-\cos 2x}{2} - 3\right) dx$
$= \frac{1}{2}\int_0^{\frac{\pi}{2}} dx - \frac{7}{2} \cdot \int_0^{\frac{\pi}{2}} \cos 2x\,dx = \frac{1}{2}x\Big|_0^{\frac{\pi}{2}} - \frac{7}{2} \cdot \frac{\sin 2x}{2}\Big|_0^{\frac{\pi}{2}} = \frac{1}{2}(\frac{\pi}{2} - 0) - \frac{7}{2}(0-0) = \frac{\pi}{4}$.

(b) $\int_0^1 \frac{dx}{\operatorname{ch} x + \operatorname{sh} x} = \int_0^1 \frac{(\operatorname{ch} x - \operatorname{sh} x)(\operatorname{ch} x + \operatorname{sh} x)}{\operatorname{ch} x + \operatorname{sh} x}\, dx = \int_0^1 (\operatorname{ch} x - \operatorname{sh} x)\, dx$
$= \int_0^1 e^{-x}\, dx = -e^{-x}\Big|_0^1 = 1 - e^{-1}.$

(c) $\int_1^8 \frac{x^4 - 4x^3 + 2\sqrt[3]{x}}{\sqrt[5]{x^4}}\, dx = \int_1^8 \frac{x^4 - 4x^3 + 2x^{\frac{1}{3}}}{x^{\frac{4}{5}}}\, dx = \int_1^2 (x^{\frac{16}{5}} - 4x^{\frac{11}{5}} + 2x^{-\frac{7}{15}})\, dx =$
$\frac{5}{21} \cdot x^{\frac{21}{5}}\Big|_1^8 - 4 \cdot \frac{5}{16} x^{\frac{16}{5}}\Big|_1^8 + 2 \cdot \frac{15}{8} x^{\frac{8}{15}}\Big|_1^8 = \cdots = -\frac{115}{42} + \frac{14395}{21 \cdot 2^{\frac{2}{5}}}.$

△

Zadatak 2.1.3. Izračunajte integrale:

$$\text{(a)} \int \operatorname{tg}^2 x\, dx \qquad \text{(b)} \int_0^{\frac{\pi}{4}} \operatorname{tg} x\, dx \qquad \text{(c)} \int_0^1 \frac{x}{x^2+1}\, dx \qquad \text{(d)} \int_{-2}^2 \lfloor x \rfloor \{x\}\, dx.$$

Rješenje. (a) $\int \operatorname{tg}^2 x\, dx = \int \frac{\sin^2 x}{\cos^2 x}\, dx = \int \frac{1-\cos^2 x}{\cos^2 x}\, dx = \int \frac{1}{\cos^2 x}\, dx - \int dx =$ $\operatorname{tg} x - x + C.$

(b) $\int_0^{\frac{\pi}{4}} \operatorname{tg} x\, dx =$ (pogađamo primitivnu funkciju) $= -\ln(\cos x)\Big|_0^{\frac{\pi}{4}} = -(\ln(\cos\frac{\pi}{4}) - \ln(\cos 0)) =$
$-(\ln\frac{1}{\sqrt{2}} - \ln 1) = \frac{1}{2}\ln 2.$

(c) $\int_0^1 \frac{x}{x^2+1}\, dx =$ (pogađamo primitivnu funkciju) $= \frac{1}{2}\ln(x^2+1)\Big|_0^1 = \frac{1}{2}(\ln 2 - \ln 1) =$
$\frac{1}{2}\ln 2.$

(d) $\int_{-2}^2 \lfloor x \rfloor \{x\}\, dx = \int_{-2}^2 \lfloor x \rfloor (x - \lfloor x \rfloor)\, dx = \int_{-2}^{-1} \lfloor x \rfloor (x - \lfloor x \rfloor)\, dx + \int_{-1}^0 \lfloor x \rfloor (x - \lfloor x \rfloor)\, dx +$
$\int_0^1 \lfloor x \rfloor (x - \lfloor x \rfloor)\, dx + \int_1^2 \lfloor x \rfloor (x - \lfloor x \rfloor)\, dx = \int_{-2}^{-1} (-2)\cdot(x+2)\, dx + \int_{-1}^0 (-1)\cdot(x+1)\, dx +$
$\int_0^1 0 \cdot x\, dx + \int_1^2 1 \cdot (x-1)\, dx = -2(\frac{x^2}{2} + 2x)\Big|_{-2}^{-1} - (\frac{x^2}{2} + x)\Big|_{-1}^0 + (\frac{x^2}{2} - x)\Big|_1^2 = \cdots = -1.$

△

Zadatak 2.1.4. Odredite integral

$$\int \sqrt{1 - \sin 2x}\, dx$$

Rješenje. Označimo $f(x) = \sqrt{1 - \sin 2x}$. Funkcija $f : \mathbb{R} \to \mathbb{R}$ je neprekidna pa po teoremu s predavanja postoji primitivna funkcija $F : \mathbb{R} \to \mathbb{R}$. Na primjer, jedna primitivna funkcija određena je formulom

$$F(x) = \int_0^x f(t)\,dt.$$

Naš cilj je naći eksplicitan izraz za funkciju F jer se svaka druga primitivna funkcija tada može zapisati u obliku $F(x) + C$ za neki $C \in \mathbb{R}$.

Koristeći identitete $1 = \cos^2 x + \sin^2 x$ i $\sin(2x) = 2\sin x \cos x$, slijedi

$$f(x) = \sqrt{\cos^2 x + \sin^2 x - 2\sin x \cos x} = \sqrt{(\cos x - \sin x)^2} = |\cos x - \sin x|.$$

Posljednji izraz možemo dodatno pojednostaviti koristeći adicijske formule:

$$\cos x - \sin x = \sqrt{2}\left(\frac{\sqrt{2}}{2}\cos x - \frac{\sqrt{2}}{2}\sin x\right) = \sqrt{2}\cos\left(x + \frac{\pi}{4}\right).$$

Dakle, integrand je jednak

$$f(x) = \sqrt{2}\left|\cos\left(x + \frac{\pi}{4}\right)\right|.$$

Funkcija $x \mapsto \cos\left(x + \frac{\pi}{4}\right)$ mijenja predznak u točkama $x_k = \frac{\pi}{4} + k\pi$ za $k \in \mathbb{Z}$. Odaberimo intervale integracije tako da im te nultočke budu rubovi, tj. označimo:

$$I_k = [x_k, x_{k+1}] = \left[\frac{\pi}{4} + k\pi, \frac{\pi}{4} + (k+1)\pi\right].$$

Za $x \in I_k$, argument $x + \frac{\pi}{4}$ poprima vrijednosti iz intervala $\left[\frac{\pi}{2} + k\pi, \frac{\pi}{2} + (k+1)\pi\right]$, na kojem funkcija kosinus ima predznak $(-1)^{k+1}$. Zato na I_k vrijedi:

$$f(x) = (-1)^{k+1}\sqrt{2}\cos\left(x + \frac{\pi}{4}\right).$$

Po Newton–Leibnizovoj formuli, za $x \in I_k$ integriramo funkciju s lijeva na desno, počevši od donje granice x_k:

$$\begin{aligned}
F(x) - F(x_k) &= \int_{x_k}^x (-1)^{k+1}\sqrt{2}\cos\left(t + \frac{\pi}{4}\right)\,dt \\
&= (-1)^{k+1}\sqrt{2}\sin\left(x + \frac{\pi}{4}\right) - (-1)^{k+1}\sqrt{2}\sin\left(x_k + \frac{\pi}{4}\right) \\
&= (-1)^{k+1}\sqrt{2}\sin\left(x_{k+1} + \frac{\pi}{4}\right) + \sqrt{2}
\end{aligned}$$

Preostaje odrediti $F(x_k)$. Uvrstimo li desni rub intervala $x = x_{k+1}$ u gornju formulu, dobivamo rekurzivnu relaciju:

$$F(x_{k+1}) - F(x_k) = (-1)^{k+1}\sqrt{2}\sin\left(x_{k+1} + \frac{\pi}{4}\right) + \sqrt{2} = 2\sqrt{2}$$

Rekurzivno slijedi:

$$F(x_k) = F(x_0) + 2k\sqrt{2}.$$

Vrijednost $F(x_0)$ možemo odrediti iz

$$F(x_0) = \int_0^{\frac{\pi}{4}} f(t)dt = \int_0^{\frac{\pi}{4}} \cos t - \sin t \, dt = \sqrt{2} - 1$$

Dakle, za $x \in I_k$ vrijedi:

$$\begin{aligned} F(x) &= (\sqrt{2} - 1) + 2k\sqrt{2} + (-1)^{k+1}(\sin x + \cos x) + \sqrt{2} \\ &= (-1)^{k+1}(\sin x + \cos x) + 2\sqrt{2}(k+1) - 1. \end{aligned}$$

Preostaje povezati k eksplicitno s varijablom x. Iz uvjeta $x \in I_k$ imamo $x_k \leq x < x_{k+1}$, što zapisujemo kao:

$$\frac{\pi}{4} + k\pi \leq x < \frac{\pi}{4} + (k+1)\pi.$$

Oduzimanjem $\frac{\pi}{4}$ i dijeljenjem cijele nejednakosti s π slijedi:

$$k \leq \frac{x}{\pi} - \frac{1}{4} < k + 1 \implies k = \left\lfloor \frac{x}{\pi} - \frac{1}{4} \right\rfloor.$$

Budući da se u našoj formuli za $F(x)$ pojavljuje izraz $k+1$, primijetimo da je

$$k + 1 = \left\lfloor \frac{x}{\pi} - \frac{1}{4} \right\rfloor + 1 = \left\lfloor \frac{x}{\pi} + \frac{3}{4} \right\rfloor.$$

Uvrštavanjem toga dobivamo konačan, egzaktan izraz na cijelom $\mathbb{R}$:

$$F(x) = (-1)^{\lfloor \frac{x}{\pi} + \frac{3}{4} \rfloor}(\sin x + \cos x) + 2\sqrt{2} \left\lfloor \frac{x}{\pi} + \frac{3}{4} \right\rfloor - 1$$

△

2.1.1 Integralne sume

Pretpostavimo da je $f : [a, b] \to \mathbb{R}$ Riemann-integrabilna funkcija. Za svako $n \in \mathbb{N}$ neka je $(x_i)_{0 \leq i \leq n}$ pripadna ekvidistantna subdivizija segmenta $[a, b]$, tj. neka je

$$h := \frac{b - a}{n} \quad \text{te} \quad x_i := a + ih, \quad \text{za} \ \ 0 \leq i \leq n.$$

Nadalje, za $n \in \mathbb{N}$ i $1 \leq i \leq n$ neka su $\xi_{n,i}$ brojevi iz segmenta $[x_{i-1}, x_i]$. Pripadna integralna suma S_n funkcije f je oblika

$$S_n = h \sum_{i=1}^{n} f(\xi_{n,i}) = \frac{b-a}{n} \sum_{i=1}^{n} f(\xi_{n,i}).$$

Tada je niz integralnih suma $(S_n)_{n\in\mathbb{N}}$ od f konvergentan i vrijedi

$$\lim_{n\to\infty} S_n = \int_a^b f(x)\, dx. \tag{2.1}$$

Ponekad je koristan i obratni proces. Naime, pretpostavimo da trebamo izračunati limes nekog niza. Ukoliko uspijemo prepoznati da su članovi tog niza u stvari integralne sume neke (Riemann integrabilne) funkcije, tada možemo upotrijebiti formulu (2.1), kako bismo našli limes tog niza.

Kao ilustraciju navodimo sljedeći zadatak.

Zadatak 2.1.5. Izračunajte limese:

(a) $\lim\limits_{n\to\infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n} \right)$

(b) $\lim\limits_{n\to\infty} \left(\frac{1}{\sqrt{4n^2-1^2}} + \frac{1}{\sqrt{4n^2-2^2}} + \ldots + \frac{1}{\sqrt{4n^2-n^2}} \right).$

Rješenje. (a) Najprije primijetimo da je

$$S_n := \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} = \frac{1}{n} \left(\frac{1}{1+\frac{1}{n}} + \frac{1}{1+\frac{2}{n}} + \cdots + \frac{1}{1+\frac{n}{n}} \right).$$

Uočimo da je S_n zapravo donja Darbouxova suma funkcije $f : [0,1] \to \mathbb{R}$ definirane formulom $f(x) := \frac{1}{1+x}$, s obzirom na n-tu ekvidistantnu subdiviziju segmenta $[0,1]$

$$x_0 := 0 < x_1 := \frac{1}{n} < x_2 := \frac{2}{n} < \cdots < x_n := \frac{n}{n} = 1$$

Naime, funkcija f je strogo padajuća na $[0,1]$, pa je

$$m_i := \min_{[x_{i-1},x_i]} f(x) = f(x_i) = \frac{1}{1+x_i} = \frac{1}{1+\frac{i}{n}}, \quad 1 \le i \le n.$$

Kako je f strogo padajuća na $[0,1]$, ona je i Riemann integrabilna na $[0,1]$, pa je prema (2.1) niz $(S_n)_{n\in\mathbb{N}}$ konvergentan i vrijedi

$$\lim_{n\to\infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \right) = \lim_{n\to\infty} S_n = \int_0^1 \frac{dx}{1+x} = \ln(1+x)\Big|_0^1 = \ln 2.$$

(b) Slično kao u (a) dijelu zadatka, najprije primijetimo da je

$$S_n := \frac{1}{\sqrt{4n^2-1^2}} + \cdots + \frac{1}{\sqrt{4n^2-n^2}} = \frac{1}{n}\left(\frac{1}{\sqrt{4-(\frac{1}{n})^2}} + \cdots + \frac{1}{\sqrt{4-(\frac{n}{n})^2}}\right).$$

Također uočimo da je S_n zapravo gornja Darbouxova suma funkcije $f : [0,1] \to \mathbb{R}$ definirane formulom $f(x) := \frac{1}{\sqrt{4-x^2}}$, s obzirom na n-tu ekvidistantnu subdiviziju segmenta $[0,1]$

$$x_0 := 0 < x_1 := \frac{1}{n} < x_2 := \frac{2}{n} < \cdots < x_n := \frac{n}{n} = 1$$

Naime, funkcija f je strogo rastuća na $[0,1]$, pa je

$$M_i := \max_{[x_{i-1},x_i]} f(x) = f(x_i) = \frac{1}{\sqrt{4-x_i^2}} = \frac{1}{\sqrt{4-(\frac{i}{n})^2}}, \quad 1 \le i \le n.$$

Kako je f strogo rasuća na $[0,1]$ ona je i Riemann integrabilna na $[0,1]$, pa je prema (2.1) niz $(S_n)_{n\in\mathbb{N}}$ konvergentan i vrijedi

$$\lim_{n\to\infty}\left(\frac{1}{\sqrt{4n^2-1^2}} + \cdots + \frac{1}{\sqrt{4n^2-n^2}}\right) = \lim_{n\to\infty} S_n = \int_0^1 \frac{dx}{\sqrt{4-x^2}} = \arcsin\frac{x}{2}\Big|_0^1 = \frac{\pi}{6}.$$

△

Zadaci za vježbu

2.1.6. Izračunajte integrale:

$$\text{(a)} \int \frac{x^2+2}{x^2+1}\,dx \qquad \text{(b)} \int (2^x+5^x)^2\,dx \qquad \text{(c)} \int \frac{dx}{1+\sin x}.$$

2.1.7. Izračunajte integrale:

$$\text{(a)} \int_0^8 (1+\sqrt{2x}+\sqrt[3]{x})\,dx \qquad \text{(b)} \int_1^4 \frac{1+\sqrt{x}}{x^2}\,dx \qquad \text{(c)} \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \frac{dx}{\sqrt{1-x^2}} \qquad \text{(d)} \int_0^{100} \lfloor x \rfloor x\,dx.$$

2.1.8. Izračunajte limese:

$$\text{(a)} \lim_{n\to\infty} \left[\frac{n}{(n+1)^2}+\ldots+\frac{n}{(2n)^2}\right] \qquad \text{(b)} \lim_{n\to\infty} \frac{1^\alpha+2^\alpha+\ldots+n^\alpha}{n^{\alpha+1}}, \quad \text{za } \alpha \geq 0$$

2.1.9. Izračunajte limese:

$$\text{(a)} \lim_{n\to\infty} \left[\frac{1}{n^2}+\frac{2}{n^2}\ldots+\frac{2n-1}{n^2}\right]$$

$$\text{(b)} \lim_{n\to\infty} \left[\frac{1}{\sqrt{2n^2+n-1^2}}+\frac{1}{\sqrt{2n^2+2n-2^2}}+\ldots+\frac{1}{\sqrt{2n^2+n\cdot n-n^2}}\right]$$

2.1.10. Izračunajte limese:

$$\text{(a)} \lim_{n\to\infty} \sqrt[n]{(1+\frac{1}{n})^{\frac{1^2}{n^2}}(1+\frac{2}{n})^{\frac{2^2}{n^2}}\cdots(1+\frac{n}{n})^{\frac{n^2}{n^2}}}$$

$$\text{(b)} \lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{k^2+4kn+5n^2}$$

2.1.11. Izračunajte

$$\int_0^1 e^x\,dx$$

koristeći integralne sume.

2.2 Metoda supstitucije i metoda parcijalne integracije

Zadatak 2.2.1. Izračunajte integrale koristeći metodu supstitucije:

$$\text{(a)} \int x^2(2x^3+4)^4\,dx \qquad \text{(b)} \int_0^{\ln 2} \frac{e^x}{\sqrt{e^x+1}}\,dx.$$

Rješenje. (a) $\displaystyle\int x^2(2x^3+4)^4\,dx = \begin{bmatrix} t = 2x^3+4 \\ dt = 6x^2\,dx \end{bmatrix} = \int \frac{t^4}{6}\,dt = \frac{t^5}{30} + C = \frac{(2x^3+4)^5}{30} + C$

(b) $$\int_0^{\ln 2} \frac{e^x}{\sqrt{e^x+1}}\,dx = \begin{bmatrix} t = e^x+1 & 0 \mapsto e^0+1 = 2 \\ dt = e^x\,dx & \ln 2 \mapsto e^{\ln 2}+1 = 3 \end{bmatrix} = \int_2^3 \frac{dt}{\sqrt{t}} = 2\sqrt{t}\,\Big|_2^3 = 2(\sqrt{3}-\sqrt{2})$$

△

Zadatak 2.2.2. Izračunajte integrale:

$$\text{(a)} \int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx \qquad \text{(b)} \int_2^3 x\sqrt{x^2-4}\,dx \qquad \text{(c)} \int \frac{x}{\sqrt{x+1}}\,dx \qquad \text{(d)} \int_0^{\frac{\pi}{2}} \cos^4 x \sin^3 x\,dx$$

$$\text{(e)} \int \sqrt{e^x-1}\,dx \qquad \text{(f)} \int_0^{\frac{\pi}{2}} \sin 2x\sqrt{1+\sin^2 x}\,dx \qquad \text{(g)}^\star \int_{-\pi}^{\pi} \frac{x}{2+\cos x}\,dx.$$

Rješenje. (a) (a) $\displaystyle\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx = \begin{bmatrix} t = \sqrt{x} \\ dt = \frac{dx}{2\sqrt{x}} \end{bmatrix} = 2\int \cos t\,dt = 2\sin t + C = 2\sin\sqrt{x} + C$

(b) $$\int_2^3 x\sqrt{x^2-4}\,dx = \begin{bmatrix} t = x^2-4 & 2 \mapsto 0 \\ dt = 2x\,dx & 3 \mapsto 5 \end{bmatrix} = \frac{1}{2}\int_0^5 \sqrt{t}\,dt = \frac{1}{3}t^{3/2}\Big|_0^5 = \frac{\sqrt{125}}{3}$$

(c) $$\int \frac{x}{\sqrt{x+1}}\,dx = \begin{bmatrix} t = \sqrt{x+1} & x = t^2-1 \\ dt = \frac{dx}{2\sqrt{x+1}} & \end{bmatrix} = 2\int (t^2-1)\,dt = \frac{2}{3}t^3 - 2t + C =$$
$$= \frac{2}{3}(x-2)\sqrt{x+1} + C$$

(d) $$\int_0^{\frac{\pi}{2}} \cos^4 x \sin^3 x\,dx = \begin{bmatrix} t = cosx & 0 \mapsto 1 \\ dt = -\sin x\,dx & \frac{\pi}{2} \mapsto 0 \end{bmatrix} = \int_0^1 t^4(1-t^2)\,dt = \left(\frac{t^5}{5} - \frac{t^7}{7}\right)\Big|_0^1 =$$
$$= \frac{2}{35}$$

(e) $\int \sqrt{e^x-1}\,dx = \left[\begin{array}{ll} t=\sqrt{e^x-1} & x=\ln(t^2+1) \\ & dx=\frac{2t\,dt}{t^2+1}\end{array}\right] = 2\int \frac{t^2}{t^2+1}\,dt =$
$= 2\int dt - 2\int \frac{dt}{t^2+1} = 2t - 2\operatorname{arctg} t + C = 2\sqrt{e^x-1} - 2\operatorname{arctg}\sqrt{e^x-1} + C$

(f) $\int\limits_0^{\frac{\pi}{2}} \sin 2x\sqrt{1+\sin^2 x}\,dx = \left[\begin{array}{ll} t=1+\sin^2 x & 0\mapsto 1 \\ dt=2\sin x\cos x\,dx & \frac{\pi}{2}\mapsto 2\end{array}\right] = \int\limits_1^2 \sqrt{t}\,dt = \frac{2}{3}t^{3/2}\Big|_1^2 =$
$= \frac{2}{3}(2\sqrt{2}-1)$

(g) $\int\limits_{-\pi}^{\pi} \frac{x}{2+\cos x}\,dx =$ neparna funkcija na simetričnoj domeni $= 0$

△

Zadatak 2.2.3. Neka je $f\colon [-a,a]\to\mathbb{R}$ Riemann-integrabilna funkcija.

(a) Ako je f neparna, dokažite da je

$$\int\limits_{-a}^{a} f(x)\,dx = 0.$$

(b) Ako je f parna, dokažite da je

$$\int\limits_{-a}^{a} f(x)\,dx = 2\int\limits_0^a f(x)\,dx.$$

Rješenje. (a) Vrijedi $\int\limits_{-a}^{a} f(x)\,dx = \left[\begin{array}{ll} t=-x & -a\mapsto a \\ dt=-dx & a\mapsto -a\end{array}\right] = \int\limits_{-a}^{a} f(-t)\,dt = -\int\limits_{-a}^{a} f(t)\,dt,$

odakle je $2\int\limits_{-a}^{a} f(x)\,dx = 0$ pa slijedi tvrdnja.

(b) $\int\limits_{-a}^{a} f(x)\,dx = \int\limits_{-a}^{0} f(x)\,dx + \int\limits_0^a f(x)\,dx = \left[\begin{array}{ll} t=-x & -a\mapsto a \\ dt=-dx & 0\mapsto 0\end{array}\right] =$
$= \int\limits_0^a f(-t)\,dt + \int\limits_0^a f(x)\,dx = \int\limits_0^a f(t)\,dt + \int\limits_0^a f(x)\,dx = 2\int\limits_0^a f(x)\,dx.$

△

Zadatak 2.2.4. Neka je $f\colon \mathbb{R} \to \mathbb{R}$ neprekidna periodična funkcija s periodom $\tau > 0$. Dokažite da za sve $a \in \mathbb{R}$ vrijedi

$$\int_a^{a+\tau} f(x)\,dx = \int_0^{\tau} f(x)\,dx.$$

Rješenje. Stavimo $m := \left\lceil \frac{a}{\tau} \right\rceil$, gdje je sa $\lceil \cdot \rceil$ označena funkcija "najmanje cijelo". Tada je $a \le m\tau < a + \tau$. Naime, iz nejednakosti

$$x \le \lceil x \rceil < x + 1, \quad \forall x \in \mathbb{R},$$

slijedi

$$a = \left(\frac{a}{\tau}\right)\tau \le m\tau = \left\lceil \frac{a}{\tau} \right\rceil \tau < \left(\frac{a}{\tau} + 1\right)\tau = a + \tau$$

pa je $a \le m\tau < a + \tau$. Sada računamo

$$\int_a^{a+\tau} f(x)\,dx = \int_a^{m\tau} f(x)\,dx + \int_{m\tau}^{a+\tau} f(x)\,dx = \begin{bmatrix} t = x - \tau & m\tau \mapsto (m-1)\tau \\ dt = dx & a + \tau \mapsto a \end{bmatrix} =$$

$$= \int_a^{m\tau} f(x)\,dx + \int_{(m-1)\tau}^{a} f(t+\tau)\,dt = \int_{(m-1)\tau}^{m\tau} f(t)\,dt =$$

$$= \begin{bmatrix} s = t - (m-1)\tau & (m-1)\tau \mapsto 0 \\ ds = dt & m\tau \mapsto \tau \end{bmatrix} = \int_0^{\tau} f(s + (m-1)\tau)\,ds = \int_0^{\tau} f(x)\,dx.$$

△

Zadatak 2.2.5. ⋆ Izračunajte integral:

$$\int_1^{1+10\pi} \max\{|\sin x|, |\cos x|\}\,dx.$$

Rješenje. Funkcija $x \mapsto \max\{|\sin x|, |\cos x|\}$ je periodična s periodom $\pi/2$ pa je

$$\int_1^{1+10\pi} \max\{|\sin x|, |\cos x|\}\,dx = \int_0^{10\pi} \max\{|\sin x|, |\cos x|\}\,dx =$$

$$= 20\int_0^{\frac{\pi}{2}} \max\{|\sin x|, |\cos x|\}\,dx = 20\left(\int_0^{\frac{\pi}{4}} \cos x\,dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin x\,dx\right) =$$

$$= 20(\sin x\Big|_0^{\frac{\pi}{4}} - \cos x\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}) = 20(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}) = 20\sqrt{2}$$

△

Vrijede formule parcijalne integracije:

$$\int u(x)\, v'(x)\, dx = u(x)v(x) - \int u'(x)v(x)\, dx \tag{2.2}$$

i

$$\int_a^b u(x)\, v'(x)\, dx = u(x)v(x)\Big|_a^b - \int_a^b u'(x)v(x)\, dx$$

Ponekad se gornje formule zapisuju kao

$$\int u\, dv = uv - \int v\, du$$

i

$$\int_a^b u\, dv = uv\Big|_a^b - \int_a^b v\, du.$$

Zadatak 2.2.6. Izračunajte integrale koristeći metodu parcijalne integracije:

$$\text{(a)} \int x\, e^x\, dx \qquad \text{(b)} \int_1^2 \frac{\ln x}{x^2}\, dx.$$

Rješenje. (a) $\displaystyle\int x\, e^x\, dx = \begin{bmatrix} u = x & du = dx \\ dv = e^x\, dx & v = e^x \end{bmatrix} = xe^x - \int e^x\, dx = e^x(x-1) + C$

(b) $$\int_1^2 \frac{\ln x}{x^2}\, dx = \begin{bmatrix} u = \ln x & du = \frac{dx}{x} \\ dv = \frac{dx}{x^2} & v = -\frac{1}{x} \end{bmatrix} = -\frac{\ln x}{x}\Big|_1^2 + \int_1^2 \frac{dx}{x^2} = -\frac{\ln 2}{2} - \frac{1}{x}\Big|_1^2 = \frac{1-\ln 2}{2}$$

△

Zadatak 2.2.7. Izračunajte integrale:

$$\text{(a)} \int_0^1 \ln(1+x^2)\, dx \qquad \text{(b)} \int_0^{\frac{\pi}{2}} e^x \sin x\, dx \qquad \text{(c)} \int \ln x\, dx \qquad \text{(d)} \int \operatorname{arctg} x\, dx.$$

Rješenje. (a) $$\int_0^1 \ln(1+x^2)\, dx = \begin{bmatrix} u = \ln(1+x^2) & du = \frac{2x dx}{1+x^2} \\ dv = dx & v = x \end{bmatrix} = x\ln(1+x^2)\Big|_0^1 - \int_0^1 \frac{2x^2 dx}{1+x^2} =$$

$$= \ln 2 - 2\int_0^1 dx + 2\int_0^1 \frac{dx}{1+x^2} = \ln 2 - 2 + 2\operatorname{arctg} x\Big|_0^1 = \ln 2 - 2 + \frac{\pi}{2}$$

(b) $\displaystyle\int_0^{\frac{\pi}{2}} e^x \sin x\, dx = \left[\begin{array}{ll} u = \sin x & du = \cos x\, dx \\ dv = e^x dx & v = e^x \end{array}\right] = e^x \sin x\Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} e^x \cos x\, dx =$

$\displaystyle = \left[\begin{array}{ll} u = \cos x & du = -\sin x\, dx \\ dv = e^x dx & v = e^x \end{array}\right] = e^{\pi/2} - e^x \cos x\Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} e^x \sin x\, dx =$

$\displaystyle = e^{\pi/2} + 1 - \int_0^{\frac{\pi}{2}} e^x \sin x\, dx \implies \int_0^{\frac{\pi}{2}} e^x \sin x\, dx = \frac{e^{\pi/2}+1}{2}$

(c) $\displaystyle\int \ln x\, dx = \left[\begin{array}{ll} u = \ln x & du = \frac{dx}{x} \\ dv = dx & v = x \end{array}\right] = x\ln x - \int x\frac{dx}{x} = x\ln x - x + C$

(d) $\displaystyle\int \operatorname{arctg} x\, dx = \left[\begin{array}{ll} u = \operatorname{arctg} x & du = \frac{dx}{x^2+1} \\ dv = dx & v = x \end{array}\right] = x \operatorname{arctg} x - \int \frac{x dx}{x^2+1} =$

$\displaystyle = x\operatorname{arctg} x - \frac{1}{2}\int \frac{d(x^2)}{x^2+1} = x\operatorname{arctg} x - \frac{1}{2}\ln(x^2+1) + C$

△

Napomena. Promotrimo sljedeći primjer

$$\begin{aligned} \int e^x \operatorname{sh} x\, dx &= \left[\begin{array}{ll} u = \operatorname{sh} x & du = \operatorname{ch} x dx \\ dv = e^x\, dx & v = e^x \end{array}\right] = e^x \operatorname{sh} x - \int e^x \operatorname{ch} x\, dx \\ &= \left[\begin{array}{ll} u = \operatorname{ch} x & du = \operatorname{sh} x dx \\ dv = e^x\, dx & v = e^x \end{array}\right] = e^x \operatorname{sh} x - \left(e^x \operatorname{ch} x - \int e^x \operatorname{sh} x\, dx\right) \\ &= e^x(\operatorname{sh} x - \operatorname{ch} x) + \int e^x \operatorname{sh} x\, dx \end{aligned}$$

Dakle, imamo

$$\int e^x \operatorname{sh} x\, dx = e^x(\operatorname{sh} x - \operatorname{ch} x) + \int e^x \operatorname{sh} x\, dx. \tag{2.3}$$

Ako s lijeve i desne strane gornje jednakosti oduzmemo $\int e^x \operatorname{sh} x\, dx$ dobit ćemo da je $e^x(\operatorname{sh} x - \operatorname{ch} x) = 0$, tj. $\operatorname{sh} x = \operatorname{ch} x$ što je očito kontradikcija. U čemu je problem?

Problem je nastupio pri doslovnoj interpretaciji formule (2.2). Naime, neka je I otvoren interval i $f : I \to \mathbb{R}$ funkcija koja ima primitivnu funkciju $F : I \to \mathbb{R}$. Neodređeni integral $\int f(x)\, dx$ funkcije f je po definiciji **klasa ekvivalencije** $F(x) + C := [F]_\sim$ funkcije F po relaciji $\sim$, gdje je $\sim$ definirana na skupu $\mathcal{D}(I)$ svih derivabilnih funkcija na I s

$$G \sim H \;:\Longleftrightarrow G' = H' \Longleftrightarrow (G-H)' = 0 \Longleftrightarrow (\exists c \in \mathbb{R})(\forall x \in I)(G(x) - H(x) = c).$$

Eksplicitno,

$$\int f(x)\,dx := F(x) + C = \{x \mapsto F(x) + c : \; c \in \mathbb{R}\}.$$

Zbog toga bi formulu (2.2) preciznije trebali zapisati u sljedećem obliku

$$\int u(x)v'(x)\,dx = (u(x)v(x) + C) - \int u'(x)v(x)\,dx.$$

Dakle, jednakost (2.3) je jednakost klasa ekvivalencija (tj. jednakost pripadnih skupova), pa jedino što iz nje možemo zaključiti je

$$e^x(\operatorname{sh} x - \operatorname{ch} x) + C = 0 + C, \quad \text{tj.}$$

$$\{x \mapsto e^x(\operatorname{sh} x - \operatorname{ch} x) + c : \; c \in \mathbb{R}\} = \{x \mapsto c : \; c \in \mathbb{R}\}.$$

Odavde specijalno slijedi da postoji konstanta $c \in \mathbb{R}$ takva da je $e^x(\operatorname{sh} x - \operatorname{ch} x) = c$, za sve $x \in \mathbb{R}$. Direktnim računom možemo provjeriti da je to uistinu tako, te da je tražena kosnstanta $c = -1$.

Zadaci za vježbu

2.2.8. Izračunajte integrale:

$$\text{(a)} \int (3x^2-2x+1)(x^3-x^2+x-9)^7\,dx \qquad \text{(b)} \int \frac{e^x+1}{e^x+x}\,dx$$

$$\text{(c)} \int_e^{e^e} x^3 e^{x^4}\,dx \qquad \text{(d)} \int_0^4 x\sqrt{x^2+9}$$

2.2.9. Izračunajte integrale:

$$\text{(a)} \int \frac{\ln x}{x^3}\,dx \qquad \text{(b)} \int \frac{x}{\cos^2 x}\,dx \qquad \text{(c)} \int e^x \sin^2 x\,dx \qquad \text{(d)} \int \frac{xe^{\operatorname{arctg} x}}{(1+x^2)^{3/2}}\,dx$$

2.2.10. Izračunajte integrale:

$$\text{(a)} \int_0^2 x^2\sqrt{4-x^2}\,dx \qquad \text{(b)} \int_0^1 \frac{x^3}{x^6+2x^3+1}\,dx \qquad \text{(c)} \int_1^e \frac{\sin\ln x}{x}\,dx$$

$$\text{(d)} \int \frac{dx}{\sin x} \qquad \text{(d)} \int \frac{dx}{\cos x}$$

2.2.11. Izračunajte integrale:

$$\text{(a)} \int \sin 3x \sin 5x\,dx \qquad \text{(b)} \int \left(\frac{\ln x}{x}\right)^2 dx \qquad \text{(c)} \int \sin\ln x\,dx \qquad \text{(d)} \int (x^4+3x)\cos x\,dx$$

2.2.12. Izračunajte integrale:

$$\text{(a)} \int \arcsin^2 x\,dx \qquad \text{(b)} \int x^2\sqrt{1+x^2}\,dx \qquad \text{(c)} \int \operatorname{arctg}\sqrt{x}\,dx \qquad \text{(d)} \int x^2 \arccos x\,dx$$

2.2.13. Izračunajte integrale:

$$\text{(a)} \int \frac{\arcsin x}{x^2}\frac{1+x^2}{\sqrt{1-x^2}}\,dx \qquad \text{(b)} \int_1^{16} \operatorname{arctg}\sqrt{\sqrt{x}-1}\,dx \qquad \text{(c)} \int e^x \sin x \sin 3x\,dx$$

2.2.14. Izračunajte integrale:

$$\text{(a)} \int \sqrt{\frac{x+\sqrt{1+x^2}}{1+x^2}}\,dx \qquad \text{(b)} \int \frac{dx}{x\ln x \ln\ln x} \qquad \text{(c)} \int \frac{\operatorname{arctg}\sqrt{x}}{\sqrt{x}}\frac{1}{1+x}\,dx$$

$$\text{(d)} \int \frac{1}{1-x^2}\ln\frac{1+x}{1-x}\,dx \qquad \text{(e)} \int \frac{dx}{e^{x/2}+e^x} \qquad \text{(f)} \int e^{x+\ln x}\,dx$$

2.2.15. Neka je $f\colon \mathbb{R} \to \mathbb{R}$ neprekidna funkcija. Dokažite sljedeće jednakosti:

(a) $\displaystyle\int\limits_a^b f(a+b-x)\,dx = \int\limits_a^b f(x)\,dx$

(b) $\displaystyle\int\limits_0^{\pi/2} f(\sin x)\,dx = \int\limits_0^{\pi/2} f(\cos x)\,dx$

(c) $\displaystyle\int\limits_0^{\pi} xf(\sin x)\,dx = \frac{\pi}{2}\int\limits_0^{\pi} f(\sin x)\,dx$

2.2.16. Izračunajte

$$\int\limits_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx.$$

2.2.17. Izračunajte

$$\lim_{t\to+\infty} \frac{\int\limits_1^t \left(\operatorname{arctg}\frac{x+1}{x-1} - \frac{\pi}{4}\right)\,dx}{\ln t}.$$

2.2.18. Dokažite

$$\lim_{t\to+\infty} \frac{\int\limits_1^{x^2} \frac{e^t}{t}\,dt}{x^{-2}e^{x^2}} = 1.$$

2.2.19. Odredite sve neprekidne funkcije $f\colon [a,b] \to [0,+\infty\rangle$ takve da je $\displaystyle\int\limits_a^b f(x)\,dx = 0$.

2.3 Integrali racionalnih funkcija

Za rješavanje integrala racionalnih funkcija koristit ćemo sljedeći poznati rezultat.

Teorem 9 (Rastav na parcijalne razlomke)**.** *Neka su p i q polinomi takvi da je*

$$q(x) = (x-x_1)^{k_1} \cdots (x-x_m)^{k_m} (x^2+p_1x+q_1)^{l_1} \cdots (x^2+p_nx+q_n)^{l_n},$$

za realne brojeve $(x_j)_{j=1}^m$, $(p_j)_{j=1}^n$, $(q_j)_{j=1}^n$ za koje vrijedi $p_j^2-4q_j<0$ i prirodne brojeve $k_1,\ldots,k_m$, $l_1,\ldots,l_n$. Tada postoji polinom P i brojevi $(A_{j,r})_{j=1,r=1}^{m,k_j}$, $(B_{j,r})_{j=1,r=1}^{n,l_j}$, $(C_{j,r})_{j=1,r=1}^{n,l_j}$ za koje vrijedi

$$\frac{p(x)}{q(x)} = P(x) + \sum_{j=1}^{m}\sum_{r=1}^{k_j} \frac{A_{j,r}}{(x-x_j)^r} + \sum_{j=1}^{n}\sum_{r=1}^{l_j} \frac{B_{j,r}x+C_{j,r}}{(x^2+p_jx+q_j)^r}.$$

Koristeći prethodni rezultat vidimo da se u teoriji, kad bismo znali faktorizirati svaki polinom, integral svake racionalne funkcije može svesti na sumu integrala polinoma te integrala oblika

$$\int \frac{dx}{(x-a)^n} \quad \text{i} \quad \int \frac{ax+b}{(x^2+px+q)^n}\,dx.$$

Prvi znamo izračunati koristeći supstituciju $x-a=t$:

$$\int \frac{dx}{(x-a)^n} = \begin{cases} \ln|x-a|+C, & n=1 \\ \frac{1}{-n+1}(x-a)^{-n+1}+C, & n\geq 1 \end{cases}.$$

Drugi integral riješit ćemo u sljedećem zadatku.

Zadatak 2.3.1. Izračunajte integrale:

(a) $\displaystyle\int \frac{dx}{x^2+a^2}$

(b) $\displaystyle\int \frac{dx}{x^2-a^2}$

(c) $\displaystyle\int \frac{ax+b}{x^2+px+q}\,dx, \quad p^2-4q<0$

Rješenje. (a)

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a^2}\int \frac{dx}{\left(\frac{x}{a}\right)^2+1} = \begin{bmatrix} t=\frac{x}{a} \\ dt=\frac{dx}{a} \end{bmatrix} = \frac{1}{a}\int \frac{dt}{t^2+1} = \frac{1}{a}\operatorname{arctg} t + C = \frac{1}{a}\operatorname{arctg}\frac{x}{a} + C$$

(b) Rastavimo izraz $\dfrac{1}{x^2-a^2}$ na parcijalne razlomke:

$$\frac{1}{x^2-a^2} = \frac{1}{(x-a)(x+a)} = \frac{A}{x-a} + \frac{B}{x+a} \implies A=\frac{1}{2a},\ B=-\frac{1}{2a}$$

Dakle,

$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\int \frac{dx}{x-a} - \frac{1}{2a}\int \frac{dx}{x+a} = \frac{1}{2a}\ln|x-a| - \frac{1}{2a}\ln|x+a| + C = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$$

(c) Najprije ćemo kvadratni trinom u nazivniku zapisati u obliku potpunog kvadrata:

$$x^2+px+q = \left(x+\frac{p}{2}\right)^2 + q - \frac{p^2}{4} = (x-\alpha)^2+\beta^2,$$

gdje je

$$\alpha = -\frac{p}{2} \quad \text{i} \quad \beta^2 = q - \frac{p^2}{4}$$

Preuređujemo brojnik $ax+b$ tako da se u njemu pojavi izraz $(x-\alpha)$:

$$ax+b = a(x-\alpha+\alpha)+b = a(x-\alpha)+a\alpha+b.$$

Uvrštavanjem modificiranog nazivnika i brojnika u početni integral dobivamo:

$$I = \int \frac{a(x-\alpha)+a\alpha+b}{(x-\alpha)^2+\beta^2}\,dx.$$

Integral rastavimo na dva dijela:

$$I = a\int \frac{x-\alpha}{(x-\alpha)^2+\beta^2}\,dx + (a\alpha+b)\int \frac{1}{(x-\alpha)^2+\beta^2}\,dx.$$

Prvi integral rješavamo jednostavnom supstitucijom $u=(x-\alpha)^2+\beta^2$, pri čemu je $du = 2(x-\alpha)\,dx$, odnosno $(x-\alpha)\,dx = \frac{du}{2}$:

$$\int \frac{x-\alpha}{(x-\alpha)^2+\beta^2}\,dx = \frac{1}{2}\int\frac{du}{u} = \frac{1}{2}\ln|u| = \frac{1}{2}\ln\big((x-\alpha)^2+\beta^2\big).$$

(Napomena: apsolutna vrijednost nije potrebna jer je $(x-\alpha)^2+\beta^2>0$ za svaki x).

Drugi integral je standardni tablični integral koji rješavamo supstitucijom $t=x-\alpha, dt = dx$:

$$\int \frac{1}{(x-\alpha)^2+\beta^2}\,dx = \frac{1}{\beta}\operatorname{arctg}\left(\frac{x-\alpha}{\beta}\right).$$

Spajanjem ova dva rješenja, integral za $n=1$ pomoću parametara α i β iznosi:

$$I = \frac{a}{2}\ln\big((x-\alpha)^2+\beta^2\big) + \frac{a\alpha+b}{\beta}\operatorname{arctg}\left(\frac{x-\alpha}{\beta}\right) + C.$$

Konačno, u originalnim parametrima p i q, izraz je jednak

$$I_1 = \frac{a}{2}\ln(x^2+px+q) + \frac{b-\frac{ap}{2}}{\sqrt{q-\frac{p^2}{4}}}\operatorname{arctg}\left(\frac{x+\frac{p}{2}}{\sqrt{q-\frac{p^2}{4}}}\right) + C.$$

△

Zadatak 2.3.2. Korištenjem rekurzivne relacije izračunajte sljedeći integral

$$\int \frac{ax+b}{(x^2+px+q)^n}, \quad p^2-4q<0,\ n\in\mathbb{N}.$$

Rješenje. Definiramo

$$I_n = \int \frac{ax+b}{(x^2+px+q)^n}\,dx, \quad n>1.$$

Ideja je brojnik $ax+b$ napisati tako da sadrži derivaciju nazivnika. Derivacija nazivnika iznosi $(x^2+px+q)' = 2x+p$. Brojnik algebarski transformiramo na sljedeći način:

$$ax+b = \frac{a}{2}(2x+p) + b - \frac{ap}{2}.$$

Uvrštavanjem ove jednakosti u početni integral dobivamo zbroj dva integrala:

$$I_n = \frac{a}{2}\int \frac{2x+p}{(x^2+px+q)^n}\,dx + \left(b-\frac{ap}{2}\right)\int \frac{1}{(x^2+px+q)^n}\,dx.$$

Prvi integral rješavamo direktnom supstitucijom $u = x^2+px+q$, odakle je $du = (2x+p)\,dx$:

$$\int \frac{2x+p}{(x^2+px+q)^n}\,dx = \int u^{-n}\,du = \frac{u^{-n+1}}{-n+1} = \frac{1}{(1-n)(x^2+px+q)^{n-1}} + C.$$

Preostaje nam riješiti drugi integral kojeg ćemo označiti sa J_n:

$$J_n = \int \frac{1}{(x^2+px+q)^n}\,dx.$$

Najprije nazivnik svodimo na potpuni kvadrat kao u prethodnom podzadatku

$$x^2+px+q = \left(x+\frac{p}{2}\right)^2 + q - \frac{p^2}{4} = (x-\alpha)^2+\beta^2,$$

gdje je $\alpha = -\frac{p}{2}$ i $\beta^2 = q-\frac{p^2}{4}$. Uz supstituciju $t = x-\alpha$ i $dt = dx$, integral postaje:

$$K_n = \int \frac{1}{(t^2+\beta^2)^n}\,dt.$$

Sada izvodimo rekurziju za niz integrala K_n. Počinjemo od zapisa za K_{n-1}:

$$\begin{aligned}
K_{n-1} &= \int \frac{1}{(t^2+\beta^2)^{n-1}}\,dt = \int \frac{t^2+\beta^2}{(t^2+\beta^2)^n}\,dt \\
&= \int \frac{t^2}{(t^2+\beta^2)^n}\,dt + \beta^2\int \frac{1}{(t^2+\beta^2)^n}\,dt \\
&= \int t\cdot\frac{t}{(t^2+\beta^2)^n}\,dt + \beta^2 K_n.
\end{aligned}$$

Prvi pribrojnik rješavamo parcijalnom integracijom. Odabiremo:

$$u = t \implies du = dt, \quad dv = \frac{t}{(t^2+\beta^2)^n}dt \implies v = \frac{(t^2+\beta^2)^{-n+1}}{2(-n+1)}.$$

Primjenom formule $\int u\,dv = u \cdot v - \int v\,du$ dobivamo:

$$\begin{aligned}\int t \cdot \frac{t}{(t^2+\beta^2)^n}\,dt &= t\frac{1}{2(1-n)(t^2+\beta^2)^{n-1}} - \int \frac{1}{2(1-n)(t^2+\beta^2)^{n-1}}\,dt\\ &= \frac{-t}{2(n-1)(t^2+\beta^2)^{n-1}} + \frac{1}{2(n-1)}K_{n-1}.\end{aligned}$$

Uvrštavanjem dobivenog izraza natrag u jednadžbu za K_{n-1} imamo:

$$K_{n-1} = \frac{-t}{2(n-1)(t^2+\beta^2)^{n-1}} + \frac{1}{2(n-1)}K_{n-1} + \beta^2 K_n.$$

Na kraju izrazimo traženi integral K_n:

$$\beta^2 K_n = K_{n-1} - \frac{1}{2(n-1)}K_{n-1} + \frac{t}{2(n-1)(t^2+\beta^2)^{n-1}}$$

$$\beta^2 K_n = \frac{2n-3}{2(n-1)}K_{n-1} + \frac{t}{2(n-1)(t^2+\beta^2)^{n-1}}.$$

Dijeljenjem s β^2 dolazimo do konačne rekurzivne formule:

$$K_n = \frac{t}{2\beta^2(n-1)(t^2+\beta^2)^{n-1}} + \frac{2n-3}{2\beta^2(n-1)}K_{n-1}.$$

Ova formula omogućuje postepeno snižavanje potencije sve do

$$K_1 = \int \frac{dt}{t^2+\beta^2} = \frac{1}{\beta}\operatorname{arctg}\frac{t}{\beta}$$

Rješenje početnog integrala I_n dobiva se vraćanjem supstitucije $t = x + \frac{p}{2}$ i zbrajanjem oba dijela. △

Zadatak 2.3.3. Izračunajte integrale:

$$\text{(a)} \int \frac{dx}{x^3-2x^2+x} \qquad \text{(b)} \int \frac{x^3dx}{x^2+x+1} \qquad \text{(c)} \int \frac{2x^3+3x^2+x-1}{(x^2+2x+2)^2}\,dx$$

$$\text{(d)} \int \frac{e^{3x}(10-2e^{3x})}{2e^{6x}-10e^{3x}+12}\,dx \qquad \text{(e)} \int \frac{\cos x}{1+\sin^4 x}\,dx.$$

Rješenje. (a) Rastav na parcijalne razlomke:

$$\frac{1}{x^3-2x^2+x}=\frac{1}{x(x-1)^2}=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{(x-1)^2}\implies A=1,\ B=-1,\ C=1$$

Odatle slijedi

$$\begin{aligned}\int\frac{dx}{x^3-2x^2+x}&=\int\frac{dx}{x}-\int\frac{dx}{x-1}+\int\frac{dx}{(x-1)^2}\\&=\ln|x|-\ln|x-1|-\frac{1}{x-1}+C\\&=\ln\left|\frac{x}{x-1}\right|-\frac{1}{x-1}+C\end{aligned}$$

(b)

$$\begin{aligned}\int\frac{x^3dx}{x^2+x+1}&=\int\left(x-1+\frac{1}{x^2+x+1}\right)dx=\frac{x^2}{2}-x+\int\frac{dx}{x^2+x+1}\\&=\frac{x^2}{2}-x+\int\frac{dx}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}=\frac{x^2}{2}-x+\frac{1}{\sqrt{\frac{3}{4}}}\operatorname{arctg}\frac{x+\frac{1}{2}}{\sqrt{\frac{3}{4}}}+C\\&=\frac{x^2}{2}-x+\frac{2}{\sqrt{3}}\operatorname{arctg}\frac{2x+1}{\sqrt{3}}+C\end{aligned}$$

(c) Rastav na parcijalne razlomke:

$$\frac{2x^3+3x^2+x-1}{(x^2+2x+2)^2}=\frac{Ax+b}{x^2+2x+2}+\frac{Cx+D}{(x^2+2x+2)^2}\implies A=2,\ B=-1,\ C=-1,\ D=1.$$

Odatle slijedi

$$\begin{aligned}&\int\frac{2x^3+3x^2+x-1}{(x^2+2x+2)^2}dx=\int\frac{2x-1}{x^2+2x+2}dx+\int\frac{-x+1}{(x^2+2x+2)^2}dx=\\&=\int\frac{2x-1}{(x+1)^2+1}dx+\int\frac{-x+1}{((x+1)^2+1)^2}dx=\begin{bmatrix}t=x+1\\dt=dx\end{bmatrix}\\&=\int\frac{2t-3}{t^2+1}dt+\int\frac{-t+2}{(t^2+1)^2}dt=\int\frac{2tdt}{t^2+1}-3\int\frac{dt}{t^2+1}-\int\frac{tdt}{(t^2+1)^2}+2\int\frac{dt}{(t^2+1)^2}\\&=\ln(t^2+1)-3\operatorname{arctg}t+\frac{1}{2(t^2+1)}+2\int\frac{dt}{(t^2+1)^2}=(*)\end{aligned}$$

Još treba izračunati:

$$\begin{aligned}\int \frac{dt}{(t^2+1)^2} &= \int \frac{1+t^2-t^2}{(t^2+1)^2}\,dt = \int \frac{dt}{t^2+1} - \int \frac{t^2dt}{(t^2+1)^2} \\ &= \operatorname{arctg} t - \int \frac{t^2dt}{(t^2+1)^2} = \begin{bmatrix} u=t & du=dt \\ dv=\frac{tdt}{(t^2+1)^2} & v=-\frac{1}{2(t^2+1)} \end{bmatrix} \\ &= \operatorname{arctg} t + \frac{t}{2(t^2+1)} - \frac{1}{2}\int \frac{dt}{t^2+1} = \operatorname{arctg} t + \frac{t}{2(t^2+1)} - \frac{1}{2}\operatorname{arctg} t + C \\ &= \frac{1}{2}\operatorname{arctg} t + \frac{t}{2(t^2+1)} + C.\end{aligned}$$

Dakle

$$\begin{aligned}(*) &= \ln(t^2+1) - 3\operatorname{arctg} t + \frac{1}{2(t^2+1)} + \operatorname{arctg} t + \frac{t}{t^2+1} + C \\ &= \ln(t^2+1) + \frac{2t+1}{2(t^2+1)} - 2\operatorname{arctg} t + C \\ &= \ln(x^2+2x+2) + \frac{2x+3}{2(x^2+2x+2)} - 2\operatorname{arctg}(x+1) + C\end{aligned}$$

(c) Najprije napravimo supstituciju

$$\int \frac{e^{3x}(10-2e^{3x})}{2e^{6x}-10e^{3x}+12}\,dx = \begin{bmatrix} t=e^{3x} \\ dt=3e^{3x}\,dx \end{bmatrix} = \frac{1}{3}\int \frac{10-2t}{2t^2-10t+12}\,dt = \frac{1}{3}\int \frac{5-t}{t^2-5t+6}\,dt = (*).$$

Integral racionalne funkcije rješavamo rastavom na parcijalne razlomke

$$\frac{5-t}{t^2-5t+6} = \frac{5-t}{(t-2)(t-3)} = \frac{A}{t-2} + \frac{B}{t-3} \implies A=-3,\ B=2.$$

Dakle

$$\begin{aligned}(*) &= -\int \frac{dt}{t-2} + \frac{2}{3}\int \frac{dt}{t-3} = -\ln|t-2| + \frac{2}{3}\ln|t-3| + C \\ &= \frac{1}{3}\ln\frac{(t-3)^2}{|t-2|^3} + C = \frac{1}{3}\ln\frac{(e^{3x}-3)^2}{|e^{3x}-2|^3} + C.\end{aligned}$$

(d)

$$\int \frac{\cos x}{1+\sin^4 x}\,dx = \begin{bmatrix} t=\sin x \\ dt=\cos x\,dx \end{bmatrix} = \int \frac{dt}{1+t^4} = (*)$$

Iz identiteta

$$t^4+1 = (t^2+1)^2 - 2t^2 = (t^2+\sqrt{2}t+1)(t^2-\sqrt{2}t+1)$$

slijedi rastav na parcijalne razlomke

$$\frac{1}{1+t^4} = \frac{At+B}{t^2+\sqrt{2}t+1} + \frac{Ct+D}{t^2-\sqrt{2}t+1} \implies A = \frac{1}{2\sqrt{2}},\ B = \frac{1}{2},\ C = -\frac{1}{2\sqrt{2}},\ D = \frac{1}{2}.$$

Odatle slijedi

$$\begin{aligned}
\int \frac{dt}{1+t^4} &= \frac{1}{2\sqrt{2}} \int \frac{t+\sqrt{2}}{t^2+\sqrt{2}t+1}\,dt - \frac{1}{2\sqrt{2}} \int \frac{t-\sqrt{2}}{t^2-\sqrt{2}t+1}\,dt = \\
&= \frac{1}{4\sqrt{2}} \int \frac{2t+\sqrt{2}}{t^2+\sqrt{2}t+1}\,dt + \frac{1}{4\sqrt{2}} \int \frac{\sqrt{2}dt}{t^2+\sqrt{2}t+1} \\
&\quad - \frac{1}{4\sqrt{2}} \int \frac{2t-\sqrt{2}}{t^2-\sqrt{2}t+1}\,dt + \frac{1}{4\sqrt{2}} \int \frac{\sqrt{2}\,dt}{t^2-\sqrt{2}t+1} \\
&= \frac{1}{4\sqrt{2}} \ln(t^2+\sqrt{2}t+1) + \frac{1}{2\sqrt{2}} \operatorname{arctg} \frac{t+\frac{\sqrt{2}}{2}}{\frac{1}{\sqrt{2}}} \\
&\quad - \frac{1}{4\sqrt{2}} \ln(t^2-\sqrt{2}t+1) + \frac{1}{2\sqrt{2}} \operatorname{arctg} \frac{t-\frac{\sqrt{2}}{2}}{\frac{1}{\sqrt{2}}} + C = \\
&= \frac{1}{4\sqrt{2}} \ln \frac{t^2+\sqrt{2}t+1}{t^2-\sqrt{2}t+1} + \frac{1}{2\sqrt{2}} \Big(\operatorname{arctg}(\sqrt{2}t+1) - \operatorname{arctg}(\sqrt{2}t-1)\Big) + C.
\end{aligned}$$

Dakle, početni integral jednak je

$$\frac{1}{4\sqrt{2}} \ln \frac{\sin^2 x+\sqrt{2}\sin x+1}{\sin^2 x-\sqrt{2}\sin x+1} + \frac{1}{2\sqrt{2}} \Big(\operatorname{arctg}(\sqrt{2}\sin x+1) - \operatorname{arctg}(\sqrt{2}\sin x-1)\Big) + C$$

△

Zadaci za vježbu

2.3.4. Izračunajte integrale:

$$\text{(a)} \int \frac{x\,dx}{x^2+x+1} \qquad \text{(b)} \int \frac{x^3dx}{x^2+2x+4} \qquad \text{(c)} \int \frac{dx}{x^4+4x^2+3}$$

2.3.5. Izračunajte integrale:

$$\text{(a)} \int \frac{(x+1)^3\,dx}{x^2-x} \qquad \text{(b)} \int \frac{dx}{x^4-1} \qquad \text{(c)} \int \frac{x^4dx}{(x+1)^3}$$

2.3.6. Izračunajte integrale:

$$\text{(a)} \int \frac{x^2dx}{(x^2+1)^3} \qquad \text{(b)} \int \frac{x^8-1}{x(x^8+1)}\,dx \qquad \text{(c)} \int \frac{x^4+1}{x^6+1}\,dx$$

2.3.7. Izračunajte integrale:

$$\text{(a)} \int \frac{dx}{(x^3+x+1)^3} \qquad \text{(b)} \int \frac{x^5+x^4-2}{x^4-4x^2+4}\,dx \qquad \text{(c)} \int \frac{x^4-1}{x(x^4-5)(x^5+5x+1)}\,dx$$