1. TAYLOROVA FORMULA

Teorem Neka f ima $n+1$ derivaciju na $\langle a, b\rangle \subseteq \mathbb{R}$, $x, 0 \in \langle a, b\rangle$. Tada postoji c_x između 0 i x tako da je

$$f(x) = f(0) + \frac{f'(0)}{1!}x + ... + \frac{f^{(n)}(0)}{n!}x^n + \frac{f^{(n+1)}(c_x)}{(n+1)!}x^{n+1}$$

Taylorov polinom stupnja n je $P_n(x) = f(0) + \frac{f'(0)}{1!}x + ... + \frac{f^{(n)}(0)}{n!}x^n$

n-ti ostatak je $R_n(x) = \frac{f^{(n+1)}(c_x)}{(n+1)!}x^{n+1}$

Dakle $f(x) = P_n(x) + R_n(x)$.

FORMULE

1) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots + (-1)^{n-1}\frac{x^n}{n} + R_n(x)$

$0 \leq x < 1 \quad |R_n(x)| \leq \frac{x^{n+1}}{n+1}$

$-1 < x < 0 \quad |R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)(1+x)}$

2) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + (-1)^{m-1}\frac{x^{2m-1}}{(2m-1)!} + R_{2m-1}(x)$

$x \in \mathbb{R} \quad |R_{2m-1}(x)| \leq \frac{|x|^{2m+1}}{(2m+1)!}$

3) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + (-1)^m\frac{x^{2m}}{(2m)!} + R_{2m}(x)$

$x \in \mathbb{R} \quad |R_{2m}(x)| \leq \frac{|x|^{2m+2}}{(2m+2)!}$

4) $e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + R_n(x)$

$x \geq 0 \quad |R_n(x)| \leq \frac{e^x x^{n+1}}{(n+1)!}$

$x < 0 \quad |R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!}$

5) $\mathrm{arctg} x = x - \frac{x^3}{3} + \frac{x^5}{5} + \ldots + (-1)^{m-1}\frac{x^{2m-1}}{2m-1} + R_{2m-1}(x)$

$x \in [-1, 1] \quad |R_{2m-1}(x)| \leq \frac{|x|^{2m+1}}{2m+1}$

6) $(1+x)^s = 1 + sx + \frac{s(s-1)}{2!}x^2 + ... + \frac{s(s-1)\cdot...\cdot(s-n+1)}{n!}x^n + R_n(x)$
$x \in \langle -1, 1], \ s \in \mathbb{R}\setminus\mathbb{N} \quad R_n(x) = \frac{s(s-1)\cdot...\cdot(s-n)}{(n+1)!}(1+c_x)^{s-n-1}x^{n+1}$

za $s - n - 1 < 0$
$(1+c_x)^{s-n-1} < (1+x)^{s-n-1}$ ako $x < 0$
$(1+c_x)^{s-n-1} < 1$ ako $x > 0$.

Posebno, za $s = -1$ znamo:

$$\frac{1}{1-x} = 1 + x + x^2 + ... + x^n + R_n(x), \quad R_n(x) = \frac{x^{n+1}}{1-x}.$$

1.1. Izračunajte Taylorov polinom stupnja 4 za:

(a) $f(x) = \ln(1 + x)$,
(b) $f(x) = \sin x$,
(c) $f(x) = \cos x$,
(d) $f(x) = e^x$,
(e) $f(x) = \operatorname{arctg} x, x \in [-1, 1]$,
(f) $f(x) = (1 + x)^s, x \in \langle -1, 1]$.

1.2. Izračunajte Taylorov polinom stupnja 3 za funkciju $f(x) = \operatorname{tg}(x)$.

1.3. Izračunajte:

(a) $\cos 0.1$ s greškom manjom od 10^{-3},
(b) $\sin 0.2$ s greškom manjom od 10^{-4}.

1.4. Izračunajte s greškom manjom od 10^{-4}:

(a) e,
(b) e^{-2},
(c) $\sqrt{e}$.

1.5. Izračunajte s greškom manjom od 10^{-3}:

(a) $\ln 1.2$,
(b) $\ln 2$,
(c) $\ln 0.9$.

1.6. Koristeći $\frac{\pi}{4} = \operatorname{arctg} 1 = \operatorname{arctg}\frac{1}{2} + \operatorname{arctg}\frac{1}{3}$ izračujte s greškom manjom od 10^{-3}.

1.7. Izračunajte $(1.2)^{\frac{1}{4}}$ s greškom manjom od 10^{-3}.

1.8. Izračunajte $\sqrt{28}$ s greškom manjom od 10^{-4}.

1.9. Izračunajte sljedeće integrale s greškom manjom od 10^{-3}:

(a) $\int_0^1 \frac{e^x - 1}{x} dx$,

(b) $\int_0^{0.1} e^{-x^2} dx$,

(c) $\int_0^1 \frac{\sin x}{x} dx$,

(d) $\int_0^1 \frac{\sin(x^2)}{x} dx$,

(e) $\int_0^{\frac{1}{2}} \frac{\cos x - 1}{x} dx$,

(f) $\int_0^1 \cos(x^2) dx$.

1.10. Koristeći prikaz funkcija pomoću Taylorovog polinoma izračunajte sljedeće limese:

(a) $\lim\limits_{x\to 0} \dfrac{\cos x - 1 + \frac{x^2}{2}}{x^3}$,

(b) $\lim\limits_{x\to 0} \dfrac{\cos x - \mathrm{e}^x}{x}$,

(c) $\lim\limits_{x\to 0} \dfrac{\ln(1+x^2)}{\sin x}$,

(d) $\lim\limits_{x\to 0} \dfrac{(1+x)^{\frac{1}{3}} - 1 - \frac{1}{3}x}{x^2}$,

(e) $\lim\limits_{x\to 0} \dfrac{e^x - e^{-x}}{x}$,

(f) $\lim\limits_{x\to 0} \dfrac{\sin x + e^x - 1}{x}$,

(g) $\lim\limits_{x\to 0} \dfrac{e^x + e^{-x} - 2}{x \sin x}$.

Rješenja

1.1. (a) $P_4(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$,

(b) $P_4(x) = x - \frac{x^3}{6}$,

(c) $P_4(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24}$,

(d) $P_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$,

(e) $P_4(x) = x - \frac{x^3}{3}$,

(f) $P_4(x) = 1 + sx + \frac{s(s-1)x^2}{2} + \frac{s(s-1)(s-2)x^3}{6} + \frac{s(s-1)(s-2)(s-3)x^4}{24}$.

1.2. $P_3(x) = x + \frac{x^3}{3}$,

1.3. (a) $\cos 0.1 = 1 - \frac{(0.1)^2}{2} + E, \ |E| < 10^{-3}$,

(b) $\sin 0.2 = 0.2^2 - \frac{(0.2)^3}{6} + E, \ |E| < 10^{-4}$,

1.4. (a) $\mathrm{e} = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{7!} + E, \ |E| < 10^{-4}$.

(b) $\mathrm{e}^{-2} = 1 + \frac{-2}{1!} + \ldots + \frac{(-2)^{10}}{10!} + E, \ |E| < 10^{-4}$,

(c) $\sqrt{\mathrm{e}} = 1 + \frac{1}{2 \cdot 1!} + \ldots + \frac{1}{2^5 \cdot 5!} + E, \ |E| < 10^{-4}$.

1.5. (a) $\ln(1.2) = 0.2 - \frac{(0.2)^2}{2} + \frac{(0.2)^3}{3} + E, \ |E| < 10^{-3}$,

(b) $\ln 2 = \ln(1 + \frac{1}{3}) + \ln(1 + \frac{1}{2}) =$
$\left(\frac{1}{3} - \frac{1}{3^2 \cdot 2} + \frac{1}{3^3 \cdot 3} - \frac{1}{3^4 \cdot 4} + \frac{1}{3^5 \cdot 5}\right) + \left(\frac{1}{2} - \frac{1}{2^2 \cdot 2} + \frac{1}{2^3 \cdot 3} + \cdots + \frac{1}{2^7 \cdot 7}\right) + E$,
$|E| < 10^{-3}$,

(c) $\ln 0.9 = -0.1 - \frac{0.1^2}{2} + E, \ |E| < 10^{-3}$.

1.6. $\pi = 4\left(\frac{1}{2} - \frac{(\frac{1}{2})^3}{3} + \ldots + \frac{(\frac{1}{2})^9}{9}\right) + 4\left(\frac{1}{3} - \frac{(\frac{1}{3})^3}{3} + \frac{(\frac{1}{3})^5}{5}\right) + E$,
$|E| < 10^{-3}$.

1.7. $(1.2)^{\frac{1}{4}} = 1 + \frac{1}{4} \cdot 0.2 + \frac{\frac{1}{4}(\frac{1}{4} - 1)}{2!}(0.2)^2 + E, \ |E| < 10^{-3}$.

1.8. $\sqrt{28} = 5\left(1 + \frac{1}{2} \cdot \frac{3}{25} + \frac{\frac{1}{2}(\frac{1}{2} - 1)}{2}(\frac{3}{25})^2 + \frac{\frac{1}{2}(\frac{1}{2} - 1)(\frac{1}{2} - 2)}{6}(\frac{3}{25})^3\right) + E$,
$|E| < 10^{-4}$.

1.9. (a) $\int_0^1 \frac{e^x-1}{x}dx = 1 + \frac{1}{2!2} + \frac{1}{3!3} + \frac{1}{4!4} + \frac{1}{5!5} + E, \ \ |E| < 10^{-3}$,

(b) $\int_0^{0.1} e^{-x^2}dx = 1 - \frac{(0.1)^3}{3} + E, \ \ |E| < 10^{-3}$,

(c) $\int_0^1 \frac{\sin x}{x}dx = 1 - \frac{1}{3\cdot 3!} + \frac{1}{5\cdot 5!} + E, \ \ |E| < 10^{-3}$,

(d) $\int_0^1 \frac{\sin(x^2)}{x}dx = \frac{1}{2} - \frac{1}{6\cdot 3!} + E, \ \ |E| < 10^{-3}$,

(e) $\int_0^{\frac{1}{2}} \frac{\cos x-1}{x}dx = -\frac{1}{2^2\cdot 2\cdot 2!} + E, \ \ |E| < 10^{-3}$,

(f) $\int_0^1 \cos(x^2)dx = 1 - \frac{1}{5\cdot 2!} + \frac{1}{9\cdot 4!} + E, \ \ |E| < 10^{-3}$.

1.10. (a) 0,

(b) -1,

(c) 0,

(d) $-\frac{1}{9}$,

(e) 2,

(f) 2,

(g) 1.