The Critical Richardson Number and the Ratio of the Eddy Transport Coefficients Obtained from a Turbulence Closure Model

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ABSTRACT

A turbulent closure model is analyzed under the condition that the turbulent flow is steady in its ensemble average and both the advection and diffusion terms, i.e., third moments of turbulence, are neglected in the turbulent Reynolds stress and heat flux equations. The critical flux Richardson number is defined as a limiting value beyond which physically correct solutions are no longer possible. All the turbulence moments are suppressed completely when the Richardson number exceeds the critical value. The validity of making such an assumption is tested against the numerical results which were obtained by utilizing a more complete set of equations.

The critical flux Richardson numbers of 0.18∼0.27 are obtained from the different proposed empirical constants. The ratio of the eddy transport coefficient of heat to that of momentum have values of 0.5∼1.2 at the critical condition of stability. A review is made to clarify the differences between the present model and the earlier works of Ellison, Townsend, and Arya.

1. Introduction

A critical condition for the transition from turbulence to laminar flow in the case of strong stratification is a controversial issue. The existence of such transition expressed in terms of the Richardson number is fairly well established from laboratory experiments (Arya and Plate, 1969; Webster, 1964) as well as field observations (Lyons *et al.* 1964; Oke, 1970).

The earlier theories have created controversial arguments concerning the behavior of the ratio of the eddy transport coefficients α (defined as K_H/K_M, where K_H and K_M are, respectively, the eddy diffusivity for heat and the eddy viscosity for momentum) and the predicted values of the critical Richardson number.

Ellison (1957) showed in his analysis that α decreased with increasing Richardson number. Townsend (1958), on the contrary, indicated that α increased with stability and that the critical gradient Richardson number was $\sim$0.08, whereas such a cutoff value of stability was not obtained by Ellison. These contradictions are due to the assumptions they were required to make on the behavior of the second moments of turbulence in order to close the set of equations for turbulence.

A turbulence closure model presented here avoids making such direct assumptions on the second moments of turbulence. But to establish a set of equations from which the second moments of turbulence can be derived, it is, of course, necessary to introduce several hypotheses to close the turbulent Reynolds stress and heat flux equations. Among these the most crucial hypothesis is that of Rotta (1951) which concerns the modeling of the correlations between the fluctuating pressure and other fluctuating variables such as wind, temperature and water vapor.[2] Other assumptions on the dissipation terms include Kolmogorov's (1941) hypothesis of local, small-scale isotropy. The third moments (triple correlations) are modeled as down-gradient type diffusion[3] terms which, however, are neglected in this paper.

2. Present model

The derivation of the present model was described in Mellor (1973, henceforth referred to as I) and was utilized as one of the three models in simulating the diurnal variation of the planetary boundary layer [Mellor and Yamada (1974), henceforth referred to as II]. Therefore, the derivation is discussed only briefly in order to indicate the modeling assumptions which have slightly more generalized forms than the ones used in I.

The most crucial assumption is that by Rotta (1951). The term

$$\overline{p\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)},$$

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[2] Rotta's original hypothesis was only for the pressure-velocity gradient correlation.

[3] See Mellor (1973) for a detailed discussion of the modeling assumptions involved.

which is referred to as the "energy redistribution term," is modeled as

$$\overline{p\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)}=-\frac{q}{3l_1}\left(\overline{u_iu_j}-\frac{\delta_{ij}}{3}q^2\right)+C_1q^2\left(\frac{\partial U_i}{\partial x_j}+\frac{\partial U_j}{\partial x_i}\right)$$
$$+C_2\beta(g_i\overline{u_j\theta}+g_j\overline{u_i\theta}-\tfrac{2}{3}\delta_{ij}g_k\overline{u_k\theta}),$$

where the overbars indicate ensemble averages, and u_i, θ and p are the parts fluctuating from the corresponding mean quantities, U_i, Θ and P, respectively; $\beta=-(\partial\rho/\partial T)_p/\rho$ is the thermal expansion coefficient, $g_j=(0,0-g)$ is the acceleration of the gravity, and $q^2=\overline{u_i^2}$ is the total turbulent kinetic energy. The length scale l_1 and the coefficients C_1 and C_2 are determined empirically. In a similar fashion

$$\overline{p\frac{\partial\theta}{\partial x_j}}=-\frac{q}{3l_2}\overline{u_j\theta}+C_3g_j\beta\overline{\theta^2},$$

where l_2 and C_3 are another length scale and a coefficient which are determined empirically (in I it was assumed that $C_2=C_3=0$).

The modeling assumptions for the dissipation terms are the same as in I. Thus it is necessary only to mention that two additional length scales, Λ_1 and Λ_2, are introduced and again they are determined empirically.

Under the condition that both advection and diffusion terms (third moments) are negligible, the resultant Reynolds stress and heat flux equations[4] (after the boundary layer approximation is applied) are:

$$\overline{w^2}:\ 2\beta g(1-\tfrac{2}{3}C_2)\overline{w\theta}-\frac{q}{3l_1}\left(\overline{w^2}-\frac{q^2}{3}\right)-\frac{2}{3}\frac{q^3}{\Lambda_1}=0 \tag{1}$$

$$\overline{uw}:\ -(\overline{w^2}-C_1q^2)\frac{\partial U}{\partial z}+\beta g(1-C_2)\overline{u\theta}-\frac{q}{3l_1}\overline{uw}=0 \tag{2}$$

$$\overline{u\theta}:\ -\overline{uw}\frac{\partial\Theta}{\partial z}-\overline{w\theta}\frac{\partial U}{\partial z}-\frac{q}{3l_2}\overline{u\theta}=0 \tag{3}$$

$$\overline{w\theta}:\ -\overline{w^2}\frac{\partial\Theta}{\partial z}+\beta g(1-C_3)\overline{\theta^2}-\frac{q}{3l_2}\overline{w\theta}=0 \tag{4}$$

$$\overline{\theta^2}:\ -\overline{w\theta}\frac{\partial\Theta}{\partial z}-\frac{q}{\Lambda_2}\overline{\theta^2}=0 \tag{5}$$

$$q^2:\ -\overline{uw}\frac{\partial U}{\partial z}+\beta g\overline{w\theta}-\frac{q^3}{\Lambda_1}=0. \tag{6}$$

For convenience, we assume that all length scales are proportional to each other, i.e.,

$$l_1,\ l_2,\ \Lambda_1,\ \Lambda_2=(A_1,A_2,B_1,B_2)l.$$

[4] $\overline{uw}$, $\overline{u\theta}$, and $\partial U/\partial z$ should be interpreted as vector quantities of stress, heat flux and wind shear, respectively.

As will be seen in the next section the expressions for the critical flux Richardson number and the ratio of the eddy transport coefficients are independent of all the length scales as long as the proportionality relations hold among them. Proportionality constants A_1, A_2, B_1, B_2 for the present analysis are assumed to be independent of the stratification. The numerical values for the empirical constants are obtained from laboratory experiments under neutral conditions as discussed in I.

All mean quantities are treated as known and the solutions are expressed in the terms of an appropriate stability parameter and q. The flux Richardson number defined as

$$\mathrm{R_f}\equiv\frac{\beta g\overline{w\theta}}{\overline{uw}(\partial U/\partial z)}, \tag{7}$$

seems the most appropriate stability parameter in the analysis.

From the turbulence energy equation (6) and the definition of the flux Richardson number (7) we obtain $\mathrm{R_f}=1-\epsilon/p$ where $\epsilon\equiv q^3/\Lambda_1$ is the dissipation term and $p\equiv-\overline{uw}(\partial U/\partial z)$ is the shear production term. There is no doubt that both ϵ and p approach zero as the stability increases. The ratio ϵ/p, however, may take a finite limiting value different from zero. To determine such a value is equivalent to predicting the critical Richardson number.

3. Critical flux Richardson number and the eddy transport coefficients

After some algebraic manipulation the solutions of the set of equations (1) to (6) may be expressed as:

$$q^2=B_1l^2\left(\frac{\partial U}{\partial z}\right)^2(1-\mathrm{R_f})\tilde{S}_M \tag{8a}$$

$$\overline{w^2}=\left[\frac{1}{3}-2\frac{A_1}{B_1}-6\frac{A_1}{B_1}(1-\tfrac{2}{3}C_2)\frac{\mathrm{R_f}}{1-\mathrm{R_f}}\right]q^2 \tag{8b}$$

$$-\overline{uw}=lq\tilde{S}_M(\partial U/\partial z) \tag{8c}$$

$$-\overline{w\theta}=lq\tilde{S}_H(\partial\Theta/\partial z) \tag{8d}$$

$$-\overline{u\theta}=-3A_2l^2(\tilde{S}_M+\tilde{S}_H)\frac{\partial\Theta}{\partial z}\frac{\partial U}{\partial z} \tag{8e}$$

$$\overline{\theta^2}=B_2l^2\tilde{S}_H(\partial\Theta/\partial z)^2. \tag{8f}$$

The terms $\tilde{S}_M$ and $\tilde{S}_H$ are given by

$$\tilde{S}_M=C_M\frac{(\mathrm{R_{fc}}-\mathrm{R_f})(\mathrm{R_{f1}}-\mathrm{R_f})}{(1-\mathrm{R_f})(\mathrm{R_{f2}}-\mathrm{R_f})}, \tag{9a}$$

$$\tilde{S}_H=C_H\frac{\mathrm{R_{fc}}-\mathrm{R_f}}{1-\mathrm{R_f}}, \tag{9b}$$

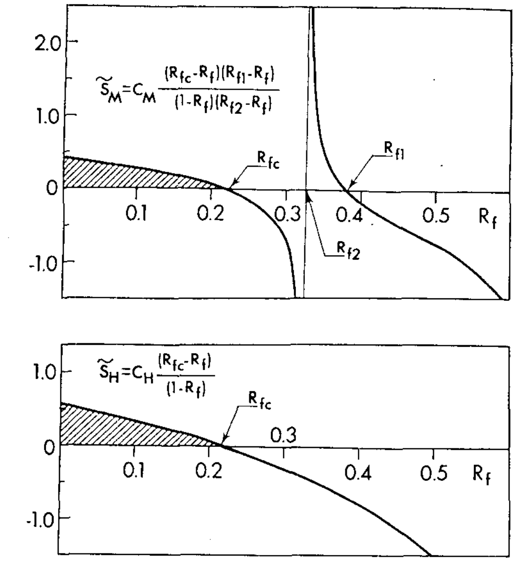

FIG. 1. Stability functions $\tilde{S}_M$ and $\tilde{S}_H$ as functions of R_f. Empirical constants are from Mellor (1973).

and R_{fc}, R_{f1}, R_{f2}, C_M, C_H are numerical constants which are determined explicitly from the previously cited empirical constants according to the following expressions:

$$R_{fc}=E_1/E_2, \tag{10a}$$

$$(R_{f1},R_{f2})=(E_3/E_4,E_1/E_5), \tag{10b,c}$$

$$C_M=\frac{A_1}{B_1}\frac{E_2E_4}{E_5}, \tag{10d}$$

$$C_H=\frac{A_2}{B_1}E_2, \tag{10e}$$

where

$$\left.\begin{aligned} E_1&=B_1-6A_1\\ E_2&=B_1+12A_1(1-C_2)+3B_2(1-C_3)\\ E_3&=B_1(1-3C_1)-6A_1\\ E_4&=B_1(1-3C_1)+12A_1(1-C_2)+9A_2(1-C_2)\\ E_5&=B_1+3A_1(1-C_2)+3B_2(1-C_3)\end{aligned}\right\}.$$

Fig. 1 shows $\tilde{S}_M$ and $\tilde{S}_H$ vs R_f. The curve of $\tilde{S}_M$ crosses the R_f axis at R_{fc} and R_{f1}. The lines $R_f=R_{f2}$ and 1.0 (not shown) are two asymptotes. On the other hand, the behavior of $\tilde{S}_H$ is much simpler than $\tilde{S}_M$ in that it decreases monotonically with increasing R_f and is zero at $R_f=R_{fc}$. The line $R_f=1.0$ (not shown) is an asymptote.

The critical value of R_f is determined as a limiting value beyond which physically correct solutions of (8) are no longer possible. The critical value (from Fig. 1) is given by (10a) which satisfies all the requirements

$$\overline{u^2}, \overline{v^2}, \overline{w^2}, \overline{\theta^2} \geqslant 0.$$

Beyond the critical value of R_f all the turbulence moments are suppressed completely. In other words, the stability functions $\tilde{S}_M$ and $\tilde{S}_H$ are assumed to be zero when R_f exceeds R_{fc} as shown in Fig. 2. In order to see the validity of making such an assumption a test is made against the numerical results which were obtained from a more complete model. The model chosen here is the one labeled as level 4 model in paper II where the hierarchy of the turbulent closure model was discussed. The present model is equivalent to the level 2 model there. The level 4 model retains the tendency terms as well as diffusion (third moments of turbulence) terms in the Reynolds stress and turbulent heat flux equations. The values of $\tilde{S}_M$ and $\tilde{S}_H$ for the level 4 model are computed diagnostically from a numerical result obtained in simulating a diurnal variation of the atmospheric boundary layer as described in II. The formulas used are

$$\tilde{S}_M=\frac{-\overline{uw}}{lq(\partial U/\partial z)}, \qquad \tilde{S}_H=\frac{-\overline{w\theta}}{lq(\partial\Theta/\partial z)}$$

which are equivalent to Eqs. (8c) and (8d), respectively. The results are shown in Fig. 2 (Mellor's values were used for the empirical constants).

The large variations in the values obtained by the level 4 numerical model in I are primarily due to the nonstationarity of the variables which are subjected to a diurnal oscillation. Additional scatters might be due to the numerical truncation errors in evaluating the vertical gradients of wind and temperature which are required to compute the stability functions $\tilde{S}_M$ and $\tilde{S}_H$.

It is interesting to note that $\tilde{S}_M$ and $\tilde{S}_H$ computed by the level 4 model did not vanish (although their values are quite small) even when the flux Richardson number exceeded its critical value determined in the present analysis. A possible explanation for this is sought in the effect of the diffusion terms (third moments) in addition to the nonstationarity of the variables as mentioned above. A recent numerical model simulation of the Wangara experiment (Clarke *et al.*, 1971) by Yamada and Mellor (1975) found that during the cooling period of a day the diffusion terms were not negligible in the balance equation for the total turbulent kinetic energy equation. Thus the turbulent energy could be transferred through the diffusive processes into the regions where the flux Richardson numbers are even greater than the critical value. Lyons *et al.* (1964) and Oke (1970) reported from the field observations that no single value could be determined for the flux Richardson number by which a distinct transition from turbulence to laminar flow occurred, although turbulence intensities decreased sharply when R_f exceeded $\frac{1}{4}$.

Thus the difference between the present stability functions and those in more complete models (or for the real atmosphere) must be taken into consideration especially when the flux Richardson number exceeds its critical value. Mellor and Durbin (1975) appear to have successfully overcome this difference by including the kinematic viscosity in their numerical modeling of the

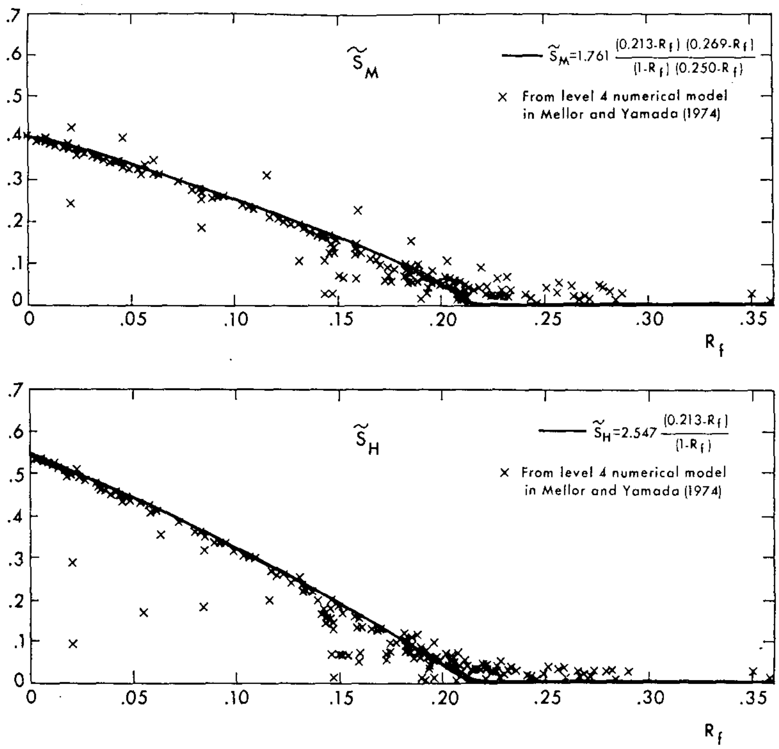

FIG. 2. Comparisons of the proposed stability functions $\tilde{S}_M$ and $\tilde{S}_H$ with those computed diagnostically from a more complete model (level 4 model) in Mellor and Yamada (1974). Empirical constants are from Mellor (1973).

oceanic thermocline where the same stability functions as presented here are utilized.

From (8c) and (8d), respectively, we have

$$K_M \equiv lq\tilde{S}_M \quad \text{and} \quad K_H \equiv lq\tilde{S}_H,$$

where K_M and K_H are eddy kinetic viscosity and eddy thermal diffusivity, respectively. It follows from the discussion given for $\tilde{S}_M$ and $\tilde{S}_H$ that K_M and K_H vanish when R_f exceeds the critical value.

Another interesting quantity is the ratio of K_H/K_M which may be expressed as

$$\alpha \equiv \frac{K_H}{K_M} = \frac{C_H}{C_M}\frac{R_{f2}-R_f}{R_{f1}-R_f}. \tag{11}$$

An important feature arises from the fact that the common factor in K_H and K_M which vanishes at $R_f=R_{fc}$ is absent in this expression. Therefore, α takes a finite value at the critical condition. This point will be considered again in discussing Ellison's model in Section 5. Fig. 3 shows α as a function of R_f where the set of constants by Mellor is utilized. Also plotted are the diagnostically computed K_H/K_M from the numerical results which are obtained by utilizing the level 4 model as already mentioned in the discussion for the stability functions $\tilde{S}_M$ and $\tilde{S}_H$.

Since $R_f=(K_H/K_M)\,R_i$ and K_H/K_M is given by Eq. (11), R_f may be expressed in terms of R_i, where R_i is the gradient Richardson number defined as

$$R_i \equiv \frac{\beta g(\partial\Theta/\partial z)}{(\partial U/\partial z)^2}.$$

If may be shown that R_f and R_i are related by a quadratic equation from which the solution may be expressed as

$$R_f = \frac{1}{2}\frac{A_2E_5}{A_1E_4}\left\{R_i + \frac{A_1E_3}{A_2E_5} - \left[R_i^2 + 2\frac{A_1}{A_2}\frac{E_3E_5-2E_1E_4}{E_5^2}R_i + \left(\frac{A_1E_3}{A_2E_5}\right)^2\right]^{\frac{1}{2}}\right\},$$

where E_1, E_3, E_4, E_5 are composite functions of empirical constants already defined in Section 3. The quantity inside the square root is shown to be positive definite when $R_{f2}<R_{f1}$ which is the present case as seen in Fig. 1.

4. Discussion on R_{fc} determined from different proposed constants

The equations presented in the previous section are by no means new. There are several models which have

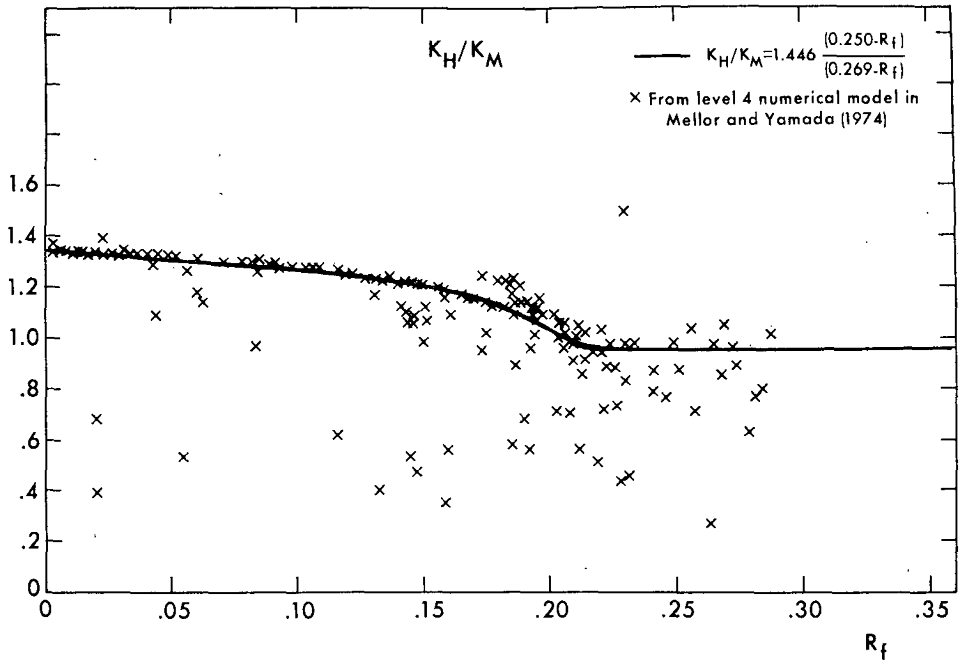

FIG. 3. The ratio K_H/K_M, of the eddy transport coefficients, as a function of R_f. Crosses are as in Fig. 2 but for α.

utilized the complete forms of Eqs. (1)–(6). This section reviews these models and discusses their differences from the present model. Donaldson (1973) derived a set of equations quite similar to that of the present analysis. However, it was not possible to relate the empirical constants of his model to the present one since Donaldson modeled the dissipation terms differently from the rest of the models discussed here. Lewellen and Teske (1973) utilized a slightly generalized form of Donaldson's model, whose length scales are related to the present ones by

$$(l_1, l_2, \Lambda_1, \Lambda_2) \Leftrightarrow \left(\frac{1}{3}, \frac{1}{3a}, \frac{1}{b}, \frac{1}{bs}\right)\lambda,$$

where the numerical values for the proportionality constants are

$$(a, b, s) = (0.75, 0.125, 1.8).$$

The next model discussed here[5] is the one proposed by Deardorff (1973). It was utilized in modeling the subgrid-scale turbulence in a three-dimensional numerical computation. He also used the complete set of Eqs. (1)–(6). A comparison of the length scales in the two models reveals the relations

$$(l_1, l_2, \Lambda_1, \Lambda_2) \Leftrightarrow \left(\frac{\sqrt{2}}{3C_M}, \frac{\sqrt{2}}{3C_s}, \frac{2\sqrt{2}}{C_E}, \frac{2\sqrt{2}}{C_\theta}\right)\Delta,$$

where Δ is the numerical grid size. The empirical values suggested by Deardorff are

$$(C_M, C_s, C_E, C_\theta) = (4.13, 4.13, 0.70, 0.42).$$

The critical flux Richardson number R_{fc}, as seen from Eq. (10a) for a closure model, can be determined explicitly from the set of empirical constants alone. Table 1 summarizes R_{fc}, and also indicates the empirical constants identified above. The model labeled "Present" in Table 1 utilized the same set of empirical constants as in Mellor for A_1, A_2, B_1, B_2. However, $C_3 = \frac{1}{3}$ was adopted after Deardorff (1973), and the value of $\frac{3}{10}$ for C_2 was determined tentatively from the analysis by Frank Lipps (GFDL Princeton, private communication). The numbers in the parentheses in Table 1 are the ratios to B_1, such as A_1/B_1, A_2/B_1, etc. It is interesting to note that R_{fc} and α depend only on the ratios (but not on the values) of the empirical constants as seen from Eqs. (10a) and (11), respectively. The evaluations of R_{fc} according to the models by Ellison, Townsend and Arya will be given in the following discussion. Their values of R_{fc} are included in Table 1 for comparison. The values of 0.2∼0.3 for R_{fc} were obtained by all models except that Townsend gave a slightly higher value of 0.5.

[5] Lumley and Khajeh-Nouri (1974) proposed a more sophisticated closure model than the one presented here. Thus it was not possible to identify the empirical constants needed to compute R_{fc} and α. Monin's model (1965) was similar to the present one except that the effect of a boundary was added to Rotta's hypothesis (but $C_1 = C_2 = C_c = 0$). Therefore, α in his model was included in Fig. 4.

Author	R_{fc}	Empirical Constants							Equation and assumptions used to obtain R_{fc}
		A_1	A_2	B_1	B_2	C_1	C_2	C_3	
Ellison (1957)	.2								Eqn.(13a) with $(T_1/T_2)\times(q^2/\overline{w^2})=4$
Townsend (1958)	.5								Eqn.(15)
Arya (1972)	.15~.4								Eqn.(17) with $F_0/F_\infty=.3\sim.6$ $G_\infty=2\sim3$
Deardorff (1973)	.18	.115 (.0284)	.115 (.0284)	4.05 (1)	6.8 (1.679)	.2	0	$\frac{1}{3}$	Eqn.(10a)
Mellor (1973)	.21	.78 (.0520)	.79 (.0527)	15 (1)	8 (.533)	.056	0	0	Eqn.(10a)
Lewellen & Teske (1973)	.24	.561 (.0417)	.747 (.056)	13.45 (1)	7.476 (.556)	0	0	0	Eqn.(10a)
Present	.27	Same as Mellor				.056	$\frac{3}{10}$	$\frac{1}{3}$	Eqn.(10a)

TABLE 1. Predicted critical flux Richardson number, R_{fc}.

5. A review of the earlier works of Ellison, Townsend and Arya

A comparison of the present closure model with the earlier works by Ellison (1957), Townsend (1958) and Arya (1972) will be given here.

Townsend utilized only Eqs. (5) and (6) in Section 2 because these equations did not require usage of Rotta's hypothesis. Ellison utilized the equation of the vertical heat flux which contained a correlation between pressure and a vertical gradient of temperature (which was included in the dissipation term). Arya (1972) adopted Rotta's hypothesis in modeling the fluctuating pressure-wind shear correlation term in the equations of $\overline{uw}$ [Eq. (2)]. A close examination of each work will follow.

Ellison (1957) utilized the vertical heat flux equations (4), the equation for temperature variance (5), and the turbulent kinetic energy equation (6) with $C_2=C_3=0$. He obtained the following expression for α:

$$\alpha=\alpha_0\frac{R_{fc}-R_f}{R_{fc}(1-R_f)^2}, \tag{12}$$

where

$$R_{fc}\equiv\left(1+\frac{T_1}{T_2}\frac{q^2}{\overline{w^2}}\right)^{-1} \tag{13a}$$

$$\alpha_0\equiv\frac{1}{2}\frac{T_3}{T_2}\frac{q^2\overline{w^2}}{(\overline{uw})^2}, \tag{13b}$$

and T_1, T_2, T_3 are dissipation time scales for $\overline{\theta^2}$, q^2, $\overline{w\theta}$ respectively; T_1 and T_2 are related to the present length scales as

$$(\Lambda_1,\Lambda_2)\Leftrightarrow 2q(T_2,T_1). \tag{14}$$

The expression (13a) must be modified, as pointed out by Arya (1972), if T_1/T_2 and/or $q^2/\overline{w^2}$ are functions of R_f. Indeed, if $q^2/\overline{w^2}$ is assumed to be given by Eq. (8b), but T_1/T_2 is assumed to be constant, it is found that (13a) is equivalent to (10a) [relation (14) is utilized to replace T's for the present length scales].

Therefore, Ellison would have obtained the same critical value as we have determined if he had treated $q^2/\overline{w^2}$ as stability dependent. The two analyses have indeed utilized the same criterion to obtain the critical value. Ellison required that $K_H\geqslant 0$ and we postulated that $\overline{u_i^2}$ and $\overline{\theta^2}$ are non-negative. The latter requires $\tilde{S}_H\geqslant 0$ which is equivalent to $K_H\geqslant 0$ from the relation $K_H=lq\tilde{S}_H$ as seen in (8d).

Ellison's analysis also differs from the present one in regard to the predicted behavior of α. Ellison's α decreases monotonically with increasing stability and vanishes at $R_f=R_{fc}$ but the present α takes a finite value of $\sim$1.0. This difference can also be eliminated if Eqs. (8b) and (8c) are substituted for $q^2\overline{w^2}/(\overline{uw})^2$ in Eq. (13b) rather than a constant as assumed by Ellison. Then it is found that α_0 has a factor $(R_{fc}-R_f)^{-1}$ which cancels out with the numerator of Eq. (12). Thus α does not vanish[6] at $R_f=R_{fc}$.

Townsend's (1958) analysis is similar to that of Ellison. However, he used only two equations: the equation of temperature variance (5) and the turbulent energy equation (6). His result is expressed as

$$R_f=\tfrac{1}{2}[1-(1-R_i/R_{ic})^{\frac{1}{2}}], \tag{15}$$

[6] According to Turner (1973, p. 150), Ellison later (1966) adopted a set similar to Eqs. (1) to (6) and obtained $\alpha=0.5$ (not zero) at $R_{fc}=0.15$.

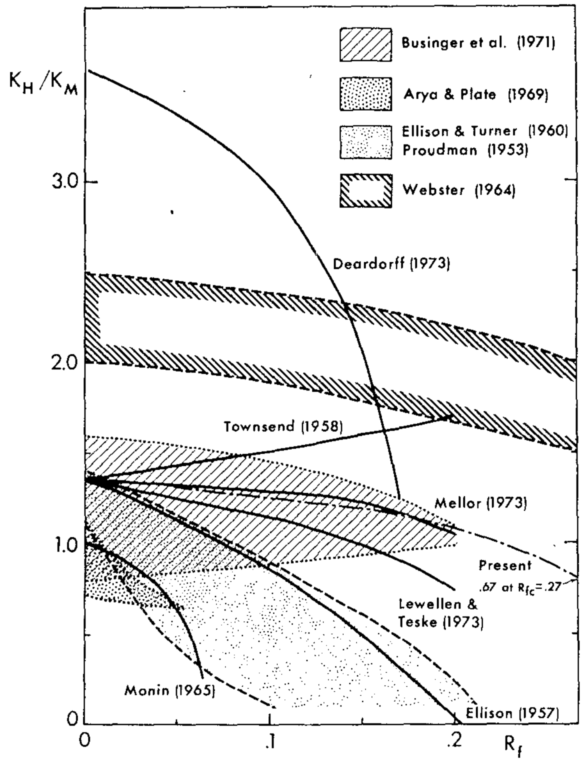

FIG. 4. A comparison of the predicted behaviors for the ratio of the eddy transport coefficients. Also shown are the results from laboratory and field observations.

where

$$R_{ic}\equiv\frac{1}{4}\frac{\Lambda_1}{\Lambda_2}\frac{\overline{\theta^2}q^2}{(\overline{w\theta})^2}\frac{(\overline{uw})^2}{q^4}.$$

Then

$$R_{fc}=\tfrac{1}{2}.$$

It is easily derived from (15) that

$$\alpha=\frac{1}{4R_{ic}(1-R_f)}. \tag{16}$$

It is clear that the analyses of both Ellison and Townsend require the assumptions on the second moments of turbulence in order to obtain a solution from the unclosed set of equations. Eq. (13a) for Ellison and Eq. (15) for Townsend are obtained by postulating that $(T_1/T_2)(q^2/\overline{w^2})$ in (13a) and R_i/R_{ic} in (15) are independent of R_f.

This point was argued by Arya (1972) from an observational point of view. He also pointed out that his wind tunnel experiments under a strong stratification indicated that K_H/K_M did not vanish as Ellison predicted but approached $\sim$0.6. Arya considered that the critical condition occurred when the mean flow failed to feed the energy to the turbulence. Since the turbulent energy production term is expressed as $-\overline{uw}\partial U/\partial z$, he considered it more appropriate to utilize the $\overline{uw}$ equation [(2)] rather than the total turbulent energy equation. His result is expressed as

$$R_f=\frac{1-(F_0/F)}{G-(F_0/F)}, \tag{17}$$

where $F\equiv\overline{w^2}q^2/(\overline{uw})^2$ and $G\equiv(\overline{u\theta}/\overline{w\theta})(\overline{uw}/\overline{w^2})$, and the zero suffix refers to the value at the neutral condition. Here F_0 is related to the present length scales as $F_0=\Lambda_1/(3l_1)$. Arya postulated that for a strong stability F and G must approach constant values according to the Monin-Obukhov similarity theory. Thus the critical flux Richardson number will be obtained from Eq. (17) if F and G are evaluated from observations. Recognizing the difficulty in obtaining the limiting values of F and G from the greatly scattered data taken under conditions of strong stratification, Arya utilized the expressions for anisotropy coefficients derived by Monin (1965) which were based on the similarity theory. After substituting the observed values for the correlations and anisotropy coefficients together with

$$\varphi_M\equiv\frac{kz}{u_\tau}\frac{\partial U}{\partial z}\sim\sigma\zeta;\ \sigma\approx 5, \tag{18}$$

Arya obtained the critical flux Richardson number of 0.15$\sim$0.40. Here $\zeta\equiv z/L$ and L is the Monin-Obukhov stability scale.

It seems that a much simpler derivation for R_{fc} is possible. For example, in the surface layer

$$R_{fc}=\lim_{\zeta\to\infty}\frac{\zeta}{\varphi_M}\sim\frac{1}{\sigma}\approx 0.2,$$

where the empirical relation (18) was used.

Predicted α's are shown in Fig. 4. Eq. (11) was utilized for the closure models; Eq. (12) with $R_{fc}=0.2$ and $\alpha_0=1.35$ resulted in Ellison's α and Eq. (16) with $R_{ic}=0.185$ produced Townsend's α. The constants for Ellison's and Townsend's models were adjusted so that their α took the value of 1.35 when $R_{fc}=0$ (neutral). Monin's result (1965) is also included in Fig. 4 for comparison even though his model is not exactly the same as the present closure model as pointed out in the footnote of Section 4.

Except for Townsend's case all the predicted α's decrease with increasing R_f which indicates that turbulent heat transfer becomes less efficient than momentum transfer in the case of strong stratification.

Fig. 4 also includes the observed values of α in the atmosphere (Businger *et al.*, 1971) in a wind tunnel (Webster, 1964; Arya and Plate) and in a water channel (Ellison and Turner, 1960). Measured values show a great scatter so that they are indicated by shaded areas.

6. Conclusions and remarks

A more complete discussion is given on the stability functions $\tilde{S}_M$ and $\tilde{S}_H$ than those given in Mellor (1973) or Mellor and Yamada (1974). It is assumed, in order to maintain the solutions to be realistic, that $\tilde{S}_M$ and $\tilde{S}_H$ vanish when the Richardson number becomes greater than the critical value. Such an assumption seems to be approximately verified by the results obtained by utilizing the more complete model described in Mellor and Yamada (1974).

A comparison is made on the ratio of the eddy transport coefficients, K_H/K_M, derived by different authors. It is not, however, possible to make rigorous comparisons of the results with observed data since the data collected here show a great scatter.

The following remarks are made from the comparison of the closure model with the earlier works. Ellison (1957) and Townsend (1958) made direct assumptions concerning the behaviors of the second moments of turbulence in order to obtain solutions. Such assumptions seem questionable, however, as indicated by the observed data taken under strong stratification which exhibit a great scatter.

On the other hand, the success of the "closure model," of course, depends on the closure assumptions and the choice of the empirical constants. Speaking of the latter, numerical values of the empirical constants are fairly easily determined from the data under the neutral stability condition as discussed in Mellor (1973). On the other hand, a direct justification of the closure assumptions from observations are very difficult since accurate measurements of the pressure fluctuation under a strong stratification are not yet available. An alternative way to examine the closure assumptions is to conduct a three-dimensional numerical simulation as done by Deardorff (1972) and make a term-by-term evaluation. The third approach is more practical, i.e., it begins by assuming the closure models are right ones and one can judge the assumptions by comparing the predictions with observations. Some success with this approach is reported by Mellor (1973), Lewellen and Teske (1973), Mellor and Yamada (1974), Wyngaard and Coté (1974), Rao *et al.* (1974) and Yamada and Mellor (1975), but it is clear that further examinations are necessary before a definite conclusion may be reached.

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