2.3 Integrali racionalnih funkcija

Zadatak 2.30 Izračunajte integrale:

$$\text{(a)} \int \frac{dx}{x^2+a^2} \qquad \text{(b)} \int \frac{dx}{x^2-a^2} \qquad \text{(c)} \int \frac{Ax+B}{x^2+px+q}\,dx,\ p^2-4q<0.$$

Rješenje.

(a) $\int \frac{dx}{x^2+a^2} = \frac{1}{a^2}\int \frac{dx}{\left(\frac{x}{a}\right)^2+1} = \begin{bmatrix} t=\frac{x}{a} \\ dt=\frac{dx}{a} \end{bmatrix} = \frac{1}{a}\int \frac{dt}{t^2+1} = \frac{1}{a}\operatorname{arctg} t + C = \frac{1}{a}\operatorname{arctg}\frac{x}{a}+C$

(b) Rastavimo izraz $\frac{1}{x^2-a^2}$ na parcijalne razlomke:
$\frac{1}{x^2-a^2} = \frac{1}{(x-a)(x+a)} = \frac{A}{x-a}+\frac{B}{x+a} \implies A=\frac{1}{2a},\ B=-\frac{1}{2a}$
Dakle,
$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\int\frac{dx}{x-a} - \frac{1}{2a}\int\frac{dx}{x+a} = \frac{1}{2a}\ln|x-a| - \frac{1}{2a}\ln|x+a| + C = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|+C$

(c) $\int \frac{Ax+B}{x^2+px+q}\,dx = \int \frac{Ax+B}{\left(x+\frac{p}{2}\right)^2+q-\frac{p^2}{4}}\,dx = \begin{bmatrix} t=x+\frac{p}{2} & x=t-\frac{p}{2} \\ dt=dx & \end{bmatrix} =$

$$= \int \frac{A(t-\frac{p}{2})+B}{t^2+q-\frac{p^2}{4}}\,dt = A\int\frac{tdt}{t^2+q-\frac{p^2}{4}} + (b-\frac{Ap}{2})\int\frac{dt}{t^2+q-\frac{p^2}{4}} =$$

$$= \frac{A}{2}\ln(t^2+q-\frac{p^2}{4}) + (B-\frac{Ap}{2})\frac{\operatorname{arctg}\frac{t}{\sqrt{q-\frac{p^2}{4}}}}{\sqrt{q-\frac{p^2}{4}}}+C =$$

$$= \frac{A}{2}\ln(x^2+px+q) + (B-\frac{Ap}{2})\frac{\operatorname{arctg}\frac{x+\frac{p}{2}}{\sqrt{q-\frac{p^2}{4}}}}{\sqrt{q-\frac{p^2}{4}}}+C$$

△

Zadatak 2.31 Izračunajte integrale:

$$\text{(a)} \int \frac{dx}{x^3-2x^2+x} \qquad \text{(b)} \int \frac{x^3dx}{x^2+x+1} \qquad \text{(c)} \int \frac{2x^3+3x^2+x-1}{(x^2+2x+2)^2}\,dx$$

$$\text{(d)} \int \frac{e^{3x}(10-2e^{3x})}{2e^{6x}-10e^{3x}+12}\,dx \qquad \text{(e)} \int \frac{\cos x}{1+\sin^4 x}\,dx.$$

(a) Rastav na parcijalne razlomke:
$\frac{1}{x^3-2x^2+x}=\frac{1}{x(x-1)^2}=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{(x-1)^2} \implies A=1,\ B=-1,\ C=1$
$\int\frac{dx}{x^3-2x^2+x}=\int\frac{dx}{x}-\int\frac{dx}{x-1}+\int\frac{dx}{(x-1)^2}=\ln|x|-\ln|x-1|-\frac{1}{x-1}+C=$
$\ln\left|\frac{x}{x-1}\right|-\frac{1}{x-1}+C$

(b) $\int\frac{x^3dx}{x^2+x+1}=\int\left(x-1+\frac{1}{x^2+x+1}\right)dx=\frac{x^2}{2}-x+\int\frac{dx}{x^2+x+1}=\frac{x^2}{2}-x+$
$\int\frac{dx}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}=\frac{x^2}{2}-x+\frac{1}{\sqrt{\frac{3}{4}}}\operatorname{arctg}\frac{x+\frac{1}{2}}{\frac{3}{4}}+C=\frac{x^2}{2}-x+\frac{2}{\sqrt{3}}\operatorname{arctg}\frac{2x+1}{\sqrt{3}}+C$

(c) Rastav na parcijalne razlomke:
$\frac{2x^3+3x^2+x-1}{(x^2+2x+2)^2}=\frac{Ax+b}{x^2+2x+2}+\frac{Cx+D}{(x^2+2x+2)^2} \implies A=2,\ B=-1,\ C=-1,\ D=1$, pa je
$\int\frac{2x^3+3x^2+x-1}{(x^2+2x+2)^2}dx=\int\frac{2x-1}{x^2+2x+2}dx+\int\frac{-x+1}{(x^2+2x+2)^2}dx=$
$=\int\frac{2x-1}{(x+1)^2+1}dx+\int\frac{-x+1}{((x+1)^2+1)^2}dx=\left[\begin{array}{ll} t=x+1 & x=t-1 \\ dt=dx & \end{array}\right]=$
$=\int\frac{2t-3}{t^2+1}dt+\int\frac{-t+2}{(t^2+1)^2}dt=\int\frac{2tdt}{t}-3\int\frac{dt}{t^2+1}-\int\frac{tdt}{(t^2+1)^2}+2\int\frac{dt}{(t^2+1)^2}=$
$\ln(t^2+1)-3\operatorname{arctg}t+\frac{1}{2(t^2+1)}+2\int\frac{dt}{(t^2+1)^2}=(*)$
Još treba izračunati:
$\int\frac{dt}{(t^2+1)^2}=\int\frac{1+t^2-1}{(t^2+1)^2}dt=\int\frac{dt}{t^2+1}-\int\frac{t^2dt}{(t^2+1)^2}=\operatorname{arctg}t-\int\frac{t^2dt}{(t^2+1)^2}=$
$\left[\begin{array}{ll} u=t & du=dt \\ dv=\frac{tdt}{(t^2+1)^2} & v=-\frac{1}{2(t^2+1)} \end{array}\right]=\operatorname{arctg}t+\frac{t}{2(t^2+1)}-\frac{1}{2}\int\frac{dt}{t^2+1}=\operatorname{arctg}t+\frac{t}{2(t^2+1)}-$
$\frac{1}{2}\operatorname{arctg}t+C=\frac{1}{2}\operatorname{arctg}t+\frac{t}{2(t^2+1)}+C$, pa je
$(*)=\ln(t^2+1)-3\operatorname{arctg}t+\frac{1}{2(t^2+1)}+\operatorname{arctg}t+\frac{t}{t^2+1}+C=\ln(t^2+1)+\frac{2t+1}{2(t^2+1)}-$
$2\operatorname{arctg}t+C=\ln(x^2+2x+2)+\frac{2x+3}{2(x^2+2x+2)}-2\operatorname{arctg}(x+1)+C$

(d) $\int\frac{e^{3x}(10-2e^{3x})}{2e^{6x}-10e^{3x}+12}dx=\left[\begin{array}{l} t=e^{3x} \\ dt=3e^{3x}\,dt \end{array}\right]=\frac{1}{3}\int\frac{10-2t}{2t^2-10t+12}dt=\frac{1}{3}\int\frac{5-t}{t^2-5t+6}dt=$
$(*)$
Rastav na parcijalne razlomke:
$\frac{5-t}{t^2-5t+6}=\frac{5-t}{(t-2)(t-3)}=\frac{A}{t-2}+\frac{B}{t-3} \implies A=-\frac{3}{5},\ B=-\frac{2}{5}$
$(*)=-\frac{1}{5}\int\frac{dt}{t-2}-\frac{2}{15}\int\frac{dt}{t-3}=-\frac{1}{5}\ln|t-2|-\frac{2}{15}\ln|t-3|+C=$

$$= \frac{1}{15} \ln \frac{(t-3)^2}{|t-2|^3} + C = \frac{1}{15} \ln \frac{(e^{3x}-1)^2}{|e^{3x}-2|^3} + C$$

(e) $\displaystyle\int \frac{\cos x}{1+\sin^4 x}\, dx = \left[\begin{array}{l} t = \sin x \\ dt = \cos x \, dx \end{array} \right] = \int \frac{dt}{1+t^4} = (*)$

Vrijedi:

$t^4 + 1 = (t^2+1)^2 - 2t^2 = (t^2 + \sqrt{2}t + 1)(t^2 - \sqrt{2}t + 1)$, odakle slijedi rastav na parcijalne razlomke:

$$\frac{1}{1+t^4} = \frac{At+B}{t^2+\sqrt{2}t+1} + \frac{Ct+D}{t^2-\sqrt{2}t+1} \implies A = \frac{1}{2\sqrt{2}},\ B = \frac{1}{2},\ C = -\frac{1}{2\sqrt{2}},\ D = \frac{1}{2}$$

$$\int \frac{dt}{1+t^4} = \frac{1}{2\sqrt{2}} \int \frac{t+\sqrt{2}}{t^2+\sqrt{2}t+1}\, dt - \frac{1}{2\sqrt{2}} \int \frac{t-\sqrt{2}}{t^2-\sqrt{2}t+1}\, dt =$$

$$= \frac{1}{4\sqrt{2}} \int \frac{2t+\sqrt{2}}{t^2+\sqrt{2}t+1}\, dt + \frac{1}{4\sqrt{2}} \int \frac{\sqrt{2}dt}{t^2+\sqrt{2}t+1} -$$

$$- \frac{1}{4\sqrt{2}} \int \frac{2t-\sqrt{2}}{t^2-\sqrt{2}t+1}\, dt + \frac{1}{4\sqrt{2}} \int \frac{\sqrt{2}\, dt}{t^2-\sqrt{2}t+1} =$$

$$= \frac{1}{4\sqrt{2}} \ln (t^2+\sqrt{2}t+1) + \frac{1}{2\sqrt{2}} \operatorname{arctg} \frac{t+\frac{\sqrt{2}}{2}}{\frac{1}{\sqrt{2}}} -$$

$$- \frac{1}{4\sqrt{2}} \ln (t^2-\sqrt{2}t+1) + \frac{1}{2\sqrt{2}} \operatorname{arctg} \frac{t-\frac{\sqrt{2}}{2}}{\frac{1}{\sqrt{2}}} + C =$$

$$= \frac{1}{4\sqrt{2}} \ln \frac{t^2+\sqrt{2}t+1}{t^2-\sqrt{2}t+1} + \frac{1}{2\sqrt{2}} \Big(\operatorname{arctg} (\sqrt{2}t+1) - \operatorname{arctg} (\sqrt{2}t-1) \Big) + C =$$

$$= \frac{1}{4\sqrt{2}} \ln \frac{\sin^2 x+\sqrt{2}\sin x+1}{\sin^2 x-\sqrt{2}\sin x+1} + \frac{1}{2\sqrt{2}} \Big(\operatorname{arctg} (\sqrt{2}\sin x+1) - \operatorname{arctg} (\sqrt{2}\sin x-1) \Big) + C$$

△

Zadaci za vježbu

2.32 Izračunajte integrale:

(a) $\displaystyle\int \frac{x\,dx}{x^2+x+1}$ (b) $\displaystyle\int \frac{x^3dx}{x^2+2x+4}$ (c) $\displaystyle\int \frac{dx}{x^4+4x^2+3}$

2.33 Izračunajte integrale:

(a) $\displaystyle\int \frac{(x+1)^3\,dx}{x^2-x}$ (b) $\displaystyle\int \frac{dx}{x^4-1}$ (c) $\displaystyle\int \frac{x^4dx}{(x+1)^3}$

2.34 Izračunajte integrale:

(a) $\displaystyle\int \frac{x^2dx}{(x^2+1)^3}$ (b) $\displaystyle\int \frac{x^8-1}{x(x^8+1)}\,dx$ (c) $\displaystyle\int \frac{x^4+1}{x^6+1}\,dx$

2.35 Izračunajte integrale:

(a) $\displaystyle\int \frac{dx}{(x^3+x+1)^3}$ (b) $\displaystyle\int \frac{x^5+x^4-2}{x^4-4x^2+4}\,dx$ (c) $\displaystyle\int \frac{x^4-1}{x(x^4-5)(x^5+5x+1)}\,dx$