Emergence of internal dynamics from a stationary quantum state

Fran Dabo¹

Mentor: dr. sc. Ryan T. Requist²

¹University of Zagreb, Faculty of Science, Bijenička Cesta 32, 10000, Zagreb, Croatia ²Ruđer Bošković Institute, Bijenička cesta 54, 10000 Zagreb, Croatia (January 21, 2025)

It is shown how time can be defined for a subsystem when it is interacting with another part (the environment) of a composite system. A variable of the environment that undergoes regular changes, e.g. the hand of a clock, is used to keep track of time. If the environment is classical, it is shown how an explicitly time-dependent Hamiltonian for the subsystem emerges from a composite system with a time-independent Hamiltonian. The interaction with the environment becomes a time-dependent potential acting upon the subsystem. The difference between the effective subsystem (now time-dependent) Schrödinger equation and the fully quantum mechanical treatment of the composite system is analysed in numerical simulations, in particular its dependence on the mass of the relevant clock variable.

We, as classical beings, think of time as something absolute, something that allows the motions of the Universe, without realising that it is actually the other way around. It is the motions of the Universe that give meaning to the concept of time.[1, 2] Time keeping used to be done by keeping track of celestial objects. The Sun's predictable movement allowed us to use the sundial as a simple clock. Later, for convenience, it was replaced by pendulum clocks. These clocks are useful to us because their motion is regular. Time is measured by keeping track of an internal coordinate of the clock. The more regular the clock, the better we can keep track of time. The ideal clock would be a point particle with constant momentum. We could then keep time with arbitrary precision by measuring its position. Equidistant points on its trajectory would correspond to isochronous time intervals.

While thinking time is necessary for the description of motion, we constructed a theory of mechanics which seems dependent on it, the notion of flowing though time was deeply ingrained in its formulation. Deeper insights into (say classical) mechanics tell us that time is not necessary for the description of a closed system.[3] Its motion is uniquely specified by a curve in phase space. If a system is open, and its energy is changing, then time can be introduced as a parameter which keeps track of the changes. We will be looking at the quantum version of this problem, which, in actuality, has a specific semi-classical character.

If the universe is thought of as a closed system, it can be described by a stationary quantum state. Such a state would be an energy eigenstate.[1] Time then would not be necessary in order to describe it as a whole. Its time evolution, a change of overall phase factor, would be unobservable. Page and Wooters argue "that the temporal behaviour we observe is actually a dependence on some internal clock time, not on an external coordinate time."[2] Time then emerges from the interaction of the universe with itself. One part of the universe serves as a clock for the other.

Consider a closed quantum system, we will call it the composite system and denote it with C. Being closed, it obeys the time-independent Schrödinger equation (TISE). Now let us divide C into two parts. One we will call the system S, and the other the environment \mathcal{E} . It is possible to find the coupled TISEs for each of these subsystems, which are together equivalent to the TISE for C.

The separation of the subsystems should not be considered exact. In fact, their interaction is crucial as it enables the parametrisation of the system in terms of environment coordinates. It will be shown how this parametrisation is responsible for the emergence of time in the system. The separation can be thought of as one between an observer and a system it is measuring. The interaction should then be kept to a minimum, as we want to be able to assign the measured properties of the system to itself alone.[4] The system influence on the environment should be near zero to avoid improper time keeping of the clock.

The TISE of S thus depends parametrically on \mathcal{E} . When kept exact, it is called the clock-dependent Schrödinger equation (CDSE).[5] The clock referring to the environment \mathcal{E} "keeping time" for the system S through the aforementioned parametrisation. The terms "environment" and "clock" will thus be used interchangeably. When the classical limit of the environment is taken, the TISE for S can be transformed into the TDSE. The requirement for a classical environment should not come as a surprise. Timedependent interaction always implies an external classical force acting on a system. Hence, a TDSE with a time-dependent potential is a mixed quantum-classical equation.[6]

1 Emergence of time in quantum mechanics

1.1 Time-independent quantum mechanics

In most textbooks the TDSE is considered the fundamental equation of quantum mechanics and the TISE is considered a special case. To show how the TISE can be thought of as more fundamental than the TDSE, we will motivate it with the time-independent Hamilton-Jacobi equation (TIHJE), following Briggs' [4] example and adjusting Cook's [7] analysis to suit the time-independent case.

We begin with the time-independent Hamilton-Jacobi equation and its accompanying momentum definition.

$$H\left(q,\frac{\partial S}{\partial q}\right) - E = 0 \tag{1}$$

$$p = \frac{\partial S}{\partial q},\tag{2}$$

where S(q) is Hamilton's characteristic function, the time-independent action, and q represents the set of conjugate coordinates. It is a function of all space, which, together with the initial conditions, can define any possible trajectory of motion permissible by the TIHJE. With this way of looking at S we will allow it to be complex and redefine it as follows:

$$S = -i\hbar \ln \psi$$
 i.e. $\psi = \exp\left(\frac{i}{\hbar}S\right)$. (3)

Now the Hamilton-Jacobi equation acquires the following form, written more explicitly,

$$H - E = T + V - E =$$

$$= \frac{1}{2m} \sum_{i} \left(\frac{\partial S}{\partial q_i} \right)^2 + V(q) - E =$$

$$= \frac{1}{2m} \sum_{i} \left| -i\hbar \frac{1}{\psi} \frac{\partial \psi}{\partial q_i} \right|^2 + V(q) - E = 0,$$
(4)

where we require the kinetic energy to be real. This is simply a redefinition of S and does not change the solution of the TIHJE. Multiplying this equation by $|\psi|^2$ we get

$$\frac{1}{2m}\sum_{i}\left|-i\hbar\frac{\partial\psi}{\partial q_{i}}\right|^{2}+|\psi|^{2}V(q)-|\psi|^{2}E=0.$$
(5)

With this form of TIHJE we can derive the TISE.

As Cook points out in [7], Schrödinger's genius was in demanding not that the Hamilton-Jacobi condition, H - E = 0, be satisfied locally for each point in space independently, but globally such that the integral

$$\int |\psi|^2 (H(\psi) - E) \,\mathrm{d}V = 0 \tag{6}$$

is true. The set of functions which satisfy this condition is a superset of the set of all solution of the TIHJE. The new condition on ψ allows it to deviate locally from the TIHJE as long as those deviations over all space cancel out.

We can now find the TISE by finding the stationary value of equation (6), which is equivalent to solving the Euler-Lagrange equations for our 'Lagrangian':

$$\mathcal{L}(\psi,\psi^*,\partial_i\psi,\partial_j\psi^*) = |\psi|^2(H-E) = \frac{\hbar^2}{2m}\sum_i \frac{\partial\psi^*}{\partial q_i}\frac{\partial\psi}{\partial q_i} + \psi^*\psi V - \psi^*\psi E.$$
(7)

Here \mathcal{L} is a functional of two independent functions ψ and ψ^* and their derivatives. Solving the Lagrange equation for ψ^* gives us the equation for ψ and vice versa

$$\frac{\partial \mathcal{L}}{\partial \psi^*} - \sum_j \frac{d}{dq_j} \frac{\partial \mathcal{L}}{\partial (\frac{\partial \psi^*}{\partial q_j})} = 0$$
(8)

$$(V - E)\psi - \frac{\hbar^2}{2m}\sum_{i}\frac{d}{dq_i}\frac{\partial\psi}{\partial q_i} = 0$$
(9)

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi \tag{10}$$

$$H\psi = E\psi \tag{11}$$

This is the TISE.

We have shown how the TISE can be derived from the TIHJE. This was done to show that the TDSE is not necessarily the fundamental equation. To see how the TISE can transform into the TDSE, we will take a look at how time can emerge if a closed quantum system is separated into smaller parts, where one of the subsystems is treated classically.

1.2 Emergence of time

Let us look at the TISE for C. The set of environment coordinates will be denoted by R and the set of system coordinates will be denoted by x. Then we can write the TISE for the composite system as

$$-\frac{\hbar^2}{2M}\frac{\partial^2\Psi}{\partial R^2} - \frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + \left[V_{\mathcal{E}}(R) + V_{\mathcal{S}}(x) + V_I(x,R) - E\right]\Psi = 0,$$
(12)

where $V_I(x, R)$ is the interaction between the subsystems. Without loss of generality[4], the total action S of the composite can be written as a sum of system and environment actions in the form

$$S(x,R) = S_{\mathcal{E}}(R) + S_{\mathcal{S}}(x,R), \tag{13}$$

which is chosen because the environment depends negligibly on system coordinates. The sum form of the action implies that the total wave-function can be written as an exact product of two wave-functions. This was shown to be true in [8, 9]. We can then write the total wave-function as

$$\Psi(x, R) = \exp\left(\frac{i}{\hbar}S(x, r)\right)$$

= $\exp\left(\frac{i}{\hbar}\left[S_{\mathcal{E}}(R) + S_{\mathcal{S}}(x, R)\right]\right)$
= $\chi(R)\psi(x, R).$ (14)

The function $|\Psi(x, R)|^2$ is a joint probability density because $\Psi(x, R)$ is a function of two coordinates. Any joint probability density can be written as the product of a marginal probability density and a conditional probability density. The same applies to probability amplitudes (wave-functions) in quantum mechanics, which is why equation (14) is true. The two amplitudes $\chi(R)$ and $\psi(x, R)$ have the following properties. [5, 10] The function

$$|\chi(R)|^2 \coloneqq \int |\Psi(x,R)|^2 \,\mathrm{d}x \tag{15}$$

is the marginal probability density of measuring the value R regardless of the value of x. While the function

$$\psi(x,R) \coloneqq \frac{\Psi(x,R)}{\chi(R)} \tag{16}$$

yields the conditional probability density $|\psi(x|R)|^2$ of measuring the value x given R was measured. It satisfies the partial normalization condition for all R

$$\int |\psi(x,R)|^2 \,\mathrm{d}x = 1. \tag{17}$$

To find the equations for each of the subsystems we substitute the product form of $\Psi(x, R)$ into equation (12) [4, 8, 10]

$$-\frac{\hbar^2}{2M}\frac{\partial^2}{\partial R^2}(\chi\psi) - \frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}\chi + \left[V_{\mathcal{E}} + V_{\mathcal{S}} + V_I - E\right]\chi\psi = 0.$$
(18)

Expanding the derivative in the first term

$$\frac{\partial^2}{\partial R^2}(\chi\psi) = \frac{\partial}{\partial R} \left(\frac{\partial\chi}{\partial R} \psi + \frac{\partial\psi}{\partial R} \chi \right)
= \frac{\partial^2\chi}{\partial R^2} \psi + 2\frac{\partial\chi}{\partial R} \frac{\partial\psi}{\partial R} + \frac{\partial^2\psi}{\partial R^2} \chi,$$
(19)

and then combining the kinetic and potential terms into their respective Hamiltonians for \mathcal{E} and \mathcal{S} alone, $H_{\mathcal{E}}$ and $H_{\mathcal{S}}$, we get

$$\psi(H_{\mathcal{E}} - E)\chi + \chi \left(H_{\mathcal{S}} + V_I - \frac{\hbar^2}{M}\frac{1}{\chi}\frac{\partial\chi}{\partial R}\frac{\partial\psi}{\partial R} - \frac{\hbar^2}{2M}\frac{\partial^2\psi}{\partial R^2}\right)\psi = 0.$$
 (20)

Multiplying this form of the TISE with ψ^* and integrating with respect to x only gives us the equation for χ

$$\left(-\frac{\hbar^2}{2M}\frac{\partial^2}{\partial R^2} + V_{\mathcal{E}}\right)\chi = (E - U_S(R))\chi \tag{21}$$

with

$$U_{S}(R) \coloneqq \int \psi^{*} \left(H_{S} + V_{I} - \frac{\hbar^{2}}{M} \frac{1}{\chi} \frac{\partial \chi}{\partial R} \frac{\partial \psi}{\partial R} - \frac{\hbar^{2}}{2M} \frac{\partial^{2} \psi}{\partial R^{2}} \right) \psi \, \mathrm{d}x.$$
(22)

This term contains the explicit interaction between S and \mathcal{E} , $V_I(x, R)$, as well as the terms with implicit interaction in the form of the derivative of $\psi(x, R)$ with respect to the environment coordinate R. Because we want the environment to function as a proper clock, it should not depend on the state of the system. This means that U_S should either be ignored or be replaced with an average constant value so that it effectively changes the energy of the environment. This means that the interaction term V_I and the parametric dependence of $\psi(x, R)$ on R should be as small as possible. In the approximation that U_S is assumed to be constant or negligible, the energy conservation for the composite no longer holds true.

The equation for the wave-function of the system ψ is

$$\left(H_{\mathcal{S}} + V_I - U_S - \frac{\hbar^2}{2M}\frac{\partial^2}{\partial R^2} - \frac{\hbar^2}{M}\frac{1}{\chi}\frac{\partial\chi}{\partial R}\frac{\partial}{\partial R}\right)\psi = 0$$
(23)

The equations (21) and (23) are exact. The proof can be found in [8]. They are coupled equations between the environment and the system. Schild calls them the clock-dependent Schrödinger equations (CDSE).[5]

The last requirement necessary to obtain the TDSE is to take the classical limit of the environment. Substituting $S_{\mathcal{E}}$ back into equation (21) and replacing $E - U_{\mathcal{S}}(R)$ with an effective E_{eff} we get The classical limit of this equation sets \hbar to zero and yields the TIHJE for \mathcal{E} .

Let us now look at the TISE for the system, equation (23). The second to last term, which contains the second derivative with respect to the environment variable, should be neglected. Its presence will not stop us from deriving a time-dependent equation, but it will be an extra term in the TDSE. This 'correction' to the TDSE will be discussed later.

The last term in equation (23) transforms into the time derivative of ψ when the classical limit for the environment is taken.

$$\frac{\hbar^2}{M} \frac{1}{\chi} \frac{\partial \chi}{\partial R} \frac{\partial \psi}{\partial R} = i\hbar \frac{1}{M} \frac{\partial S_{\mathcal{E}}}{\partial R} \frac{\partial \psi}{\partial R}
= i\hbar \frac{\partial R}{\partial t} \frac{\partial \psi}{\partial R}
= i\hbar \frac{\partial \psi}{\partial t}.$$
(25)

The derivative of the environment action was identified as the classical momentum, which leads to the time derivative of the wave-function. The dependency of the system on R was replaced with the dependence on t(R). This shows clearly how the environment coordinate serves to keep time for the system. The same time that is used to describe the now classical environment is used to describe the quantum system. Equation(23) can now be written as

$$\left(H_{\mathcal{S}} + V_I(x,t) - U_S(t) - i\hbar\frac{\partial}{\partial t}\right)\psi = 0$$
(26)

Since the potential U_S is independent of the system variables, it can be removed from equation (26) with a phase change [6]

$$\psi = \exp\left(\frac{i}{\hbar} \int_{-\infty}^{t} U_S(t') \,\mathrm{d}t'\right) \widetilde{\psi}.$$
(27)

This finally yields the TDSE for the system.

$$\left(H_{\mathcal{S}} + V_I(x,t) - i\hbar\frac{\partial}{\partial t}\right)\psi = 0$$
(28)

The only dependence on time in this equation is in $V_I(x,t)$. This clearly shows that the time-dependent part of the potential stems from the interaction with an external system. We also know that this external system is a classical one, for example, a classical electromagnetic field used to observe a system. This is why the TDSE with a time-dependent Hamiltonian is a semi-classical equation. A complete description of an environment interacting with a system requires both to be described fully quantum mechanically with a single wave-function.

2 Leading corrections to the effective TDSE

In deriving the TDSE for the system S we decided to ignore the term $-\frac{\hbar^2}{2M}\frac{\partial^2\psi}{\partial R^2}$ in equation (23). The term is second order in \hbar making it a small correction. If we assume the perfect clock, a free classical particle, R = vt holds true. Then we can transform the correction term into $-\frac{\hbar^2}{2Mv^2}\frac{\partial^2\psi}{\partial t^2}$. The classical kinetic energy in the denominator makes the term even smaller. For a less ideal clock, say an observer, the dependence of t on R would be complicated, if not impossible to calculate, making the second derivative on ψ a mystery operator.

The picture of a system being influenced by a classical environment can be applied to electrons and nuclei in solids and molecules. It can be a good approximation to treat nuclei as classical particles as they can be several orders of magnitude heavier then the electrons. Here, we consider a model closed quantum system representing this situation. From it we can try to find some properties of the difference from the corresponding TDSE. The model system will consist of two coupled harmonic oscillators. One with a light mass will represent the system S and one with a heavy mass the environment \mathcal{E} . We will explore how, when a classical limit for the heavy oscillator is taken, a classical time-dependent driving potential emerges for the light oscillator. We will further quantify how the leading order correction to this classical approximation depends on the heavy oscillator mass.

We choose the Hamiltonian for the model system to be

$$H = \frac{1}{2}(p^2 + q^2) + \frac{1}{2M}P^2 + \frac{1}{2}Q^2 + \alpha q^2 Q.$$
 (29)

P, Q and p, q are the impulse and position of the 'large' and 'small' oscillator, respectively. The small oscillator will play the role of the system S, and the large oscillator will serve as the environment \mathcal{E} . The Hamiltonian depends on two parameters, the mass of the large oscillator M and the strength of the oscillators interaction α . To completely define the model system we must also specify the initial state. The initial state we chose for this model is the ground state of the 'shifted oscillator', $|\psi_0^0\rangle$, whose Hamiltonian is

$$H^{0} = \frac{1}{2}(p^{2} + q^{2}) + \frac{1}{2M}P^{2} + \frac{1}{2}(Q - Q_{0})^{2} + \alpha q^{2}(Q - Q_{0}).$$
(30)

Q0 is the third and final parameter, the 'starting position' for the large oscillator. To explain this choice of initial state, let us look at the Hamiltonian for the environment alone H_{env} and its shifted version H_{env}^0

$$H_{env} = \frac{1}{2M}P^2 + \frac{1}{2}Q^2 \tag{31}$$

$$H_{env}^{0} = \frac{1}{2M}P^{2} + \frac{1}{2}(Q - Q_{0})^{2}.$$
(32)

The ground state of H_{env}^0 , $|\psi_{env,0}^0\rangle$, can be expanded in terms of H_{env} eigenstates, $|\psi_n\rangle$, where n is the oscillator quantum number. We can now examine the time evolution of the initial state $|\psi_{env,0}^0\rangle$ under the influence of H_{env} .

$$\begin{aligned} |\psi_{env,0}^{0}(t)\rangle &= e^{-iH_{env}t/\hbar} |\psi_{env,0}^{0}\rangle \\ &= \sum_{n=0}^{\infty} c_{n} |\psi_{n}\rangle e^{-iE_{n}t/\hbar}, \ c_{n} &= \langle \psi_{n} |\psi_{env,0}^{0}\rangle \end{aligned}$$
(33)

In particular we are interested in how the expectation value $\langle Q \rangle$ changes over time. The time dependence is equal to

$$\langle Q \rangle(t) = \langle \psi^0_{env,0}(t) | Q | \psi^0_{env,0}(t) \rangle = Q_0 \cos(\Omega t).$$
(34)

It oscillates with the frequency of the large oscillator $\Omega = 1/\sqrt{M}$.

If we were to to take the classical limit of the large oscillator, the momentum and position in H would stop being operators and would become numbers equal to their expectation values. As the initial state is the ground state of H^0 , H_{env} and the position operator in the interaction term should be replaced with their expectation values.

$$H_{env} \longrightarrow \frac{\hbar\Omega}{2} + \frac{Q_0^2}{2}$$
$$\alpha q^2 Q \longrightarrow \alpha Q_0 q^2 \cos(\Omega t)$$

What we are left with is a time-dependent Hamiltonian which corresponds to the model system if the environment was classical

$$H(t) = \frac{1}{2}(p^2 + q^2) + \frac{\hbar\Omega}{2} + \frac{Q_0^2}{2} + \alpha Q_0 q^2 \cos(\Omega t)$$

$$= \frac{1}{2}p^2 + \frac{1}{2}\left[1 + 2\alpha Q_0 \cos(\Omega t)\right]q^2 + \frac{\hbar\Omega}{2} + \frac{Q_0^2}{2}$$

$$= \frac{1}{2}p^2 + \frac{1}{2}\left[1 + 2\alpha Q_0 \cos\left(\frac{t}{\sqrt{M}}\right)\right]q^2 + \frac{\hbar}{2\sqrt{M}} + \frac{Q_0^2}{2}$$
(35)



Figure 1: (a) The energies at t=0 of the light subsystem and the classically driven analogue for different values of mass M.

(b) The differences of the energies in (a) as well as the linear fit to the data. The slope of the fitted line is approximately -0.5.

After subtracting the classical energy of the heavy oscillator $Q_0^2/2$ from H(t) we are left with the Hamiltonian of the classically driven analogue of the model system. It is a harmonic oscillator Hamiltonian with a shift in energy whose frequency changes over time. A system described by a time dependent hamiltonian requires the TDSE to solve.

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \left[\frac{1}{2}p^2 + \frac{1}{2}\left[1 + 2\alpha Q_0 \cos(\Omega t)\right]q^2 + \frac{\hbar\Omega}{2}\right]|\psi(t)\rangle \tag{36}$$

Now that we know what the classically driven analogue of the model system is, we can compare them and see how their difference depends on system parameters. We will focus on finding how it depends on the environment mass M up to the leading order. We expect an inverse mass dependence as the model system and its classically driven analogue should become more similar as the environment in the model becomes more classical, i.e. its mass increases.

The property of the systems we will be comparing is their energy. The energy of the classically driven analogue varies over time because it is being influenced by an external environment. However, the energy of the model system is constant in time because it is a closed system. We must compare only a subsystem of the model to its classically driven analogue. The classically driven analogue Hamiltonian does not include the environment coordinates, only the ground state expectation values of them. It does not describe the environment. Because of that, the energy of the model which corresponds to the classical analogue energy is the expectation value of the system Hamiltonian plus the interaction term, $H_{S} + V_{I}$.

Light subsystem:
$$E^{LS}(t) = \langle \psi_0^0(t) | \frac{1}{2} (p^2 + q^2) + \alpha q^2 Q | \psi_0^0(t) \rangle$$
 (37)

Classically driven analogue:
$$E^{CDA}(t) = \langle \psi(t) | \frac{1}{2}p^2 + \frac{1}{2} [1 + 2\alpha Q_0 \cos(\Omega t)] q^2 + \frac{\hbar\Omega}{2} |\psi(t)\rangle.$$
 (38)

The wave-function $|\psi_0^0(t)\rangle$ is the time evolution of $|\psi_0^0\rangle$ under the influence of H.

For the purpose of the simulations, the values of the small oscillator mass and \hbar were chosen to be one, $m = \hbar = 1$. The values of the interaction strength and the starting position of the large oscillator are $\alpha = 0.01$ and $Q_0 = 3$. The simulation of the classically driven analogue system utilised the Crank–Nicolson method. The range of the spatial domain was $q \in [-5, 5]$ with 20000 grid points. The value of the time step was varied to accommodate simulation time and accuracy. The simulation of the model system used finite matrices and vectors to approximate the $H_S + V_I$ operator and the initial state.

Before taking a look at how the two systems evolve, we must make sure their initial states are analogous. For the initial state of the model system, the ground state of H^0 , $|\psi_0^0\rangle$, the expectation value



Figure 2: A graph of E^{LS} and E^{CDA} over time as well as their difference for M = 1000.

 $\langle Q \rangle$ is equal to Q_0 , the maximum possible. Therefore, the initial state of the classically driven analogue should be the one for which $\alpha Q_0 \cos(\Omega t)$ is maximum. That is simply the ground state of a quantum oscillator with frequency $\sqrt{1 + 2\alpha\Omega}$.

It is possible to compare the systems even without looking at their time evolution by comparing their initial state energies, $E^{LS}(t=0)$ and $E^{CDA}(t=0)$. Figure 1(a) shows the two energies as a function of the large oscillator mass M. As expected, as M increases their difference decreases. Figure 1(b) shows the differences of the two energies as a function of M together with a linear fit to the logarithm of the data. The linear fit slope is equal to $a = -0.5298 \pm 0.0057$. We do not expect this result to be exact up to the standard deviation as it is a result of simulations. The calculated slope implies that the energy difference changes as $M^{-0.5}$.

Figure 2 shows an example of the time evolution of E^{LS} and E^{CDA} and their difference for M = 1000. It oscillates with two distinct frequencies. The wavelength of the high frequency oscillations does not change with M. They are an artifact of the small oscillator. The wavelength of the low frequency oscillations increases with increasing M.



Figure 3: Energy differences for several M plotted in blue. On top is a double moving average of the data in orange. Red dots denote some of the maxima of the moving averages. The green lines are fitted to the shown maxima.

Figure 3 shows the energy differences for a range of masses over a longer period of time. It can be seen that the average value of the differences decreases with increasing mass. It is also apparent that the



Figure 4: (a) The dependence of the slopes of the fitted lines in figure 3 on mass M. It shows that the increase of the energy difference grows with $M^{-0.5}$.

(b) The dependence of the average values of the energy differences in figure 3 on mass M. It shows a very convincing dependence on $M^{-0.5}$.

amplitude of the longer wavelength oscillations increases over time, while the amplitude of the shorter wavelength oscillations decreases with increasing mass. The increase of the longer wavelength amplitude over time is the result of the classically driven analogue having a time-dependent Hamiltonian. The time-dependent part causes it to gain energy over time. The slope of the envelope of the oscillations is a measure of how quickly the classically driven analogue energy increases. We can find how it depends on mass M. First, to remove the short wavelength oscillations, a double moving average of the data is taken. Then the local maxima of the longer wavelength oscillations are found and a line is fitted do them. Because we are focusing on the asymptotic behaviour at long times, we will use only the latter half of the maxima in the fitting to avoid any short-time transient behaviour.

Figure 4(a) shows how the slope of the fitted lines in figure 3 depends on M. It indicates, together with figure 3, that the increase of energy difference is proportional to $\sim tM^{-0.5}$. The fact that both of the exponents of M calculated so far are close to -0.5 is not unexpected. Based on estimates of mass dependent perturbations of time-independent systems we expect that the leading order correction to the classically driven analogue energy scales as $M^{-0.5}$. We believe that the form of this correction could be derived by an asymptotic analysis of the system. Figure 4(b) shows how the average values of the energy differences seen in figure 3 depend on M. It is analogous to figure 1(b) and reinforces its result with a value of the exponent of M even closer to -0.5.

3 Conclusion

An argument was made that time is not something inherent to motion. Instead, time-keeping is a result of comparing the motion of one object to another. With that kind of thinking, we put timedependent mechanics aside and, using the TIHJE, we motivated the TISE. Time-independent quantum mechanics is exact for a closed system. However, if we wish to measure quantum systems, interaction is inevitable. To model this behaviour, a closed quantum system is artificially separated into two interacting parts. It can be shown how the TISE for the closed system can be transformed into the TDSE of one of the subsystems. In this scenario, one of the subsystems acts as a clock that keeps track of time for the other. It is important that their interaction is negligible for the clock to function properly. A time variable emerges when a classical limit of the clock system is taken.

The approximation of a classical clock means that the resulting TDSE is also approximate. To find how the TISE and its analogous TDSE differ, we looked at a model system of two interacting quantum oscillators. Numerical simulations show that, up to the leading order, the difference of energies decreases with the square root of the clock mass. Looking at the time evolution of the energy differences, it can be seen that they oscillate with a wavelength that increases with the clock mass. The amplitude of the oscillations, corresponding to the energy gained because of the time-dependent potential, increases linearly in time and decreases with the square root of the clock mass.

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