1.2. LINEARNA KOMBINACIJA VEKTORA LINEARNA NEZAVISNOST VEKTORA

Vektorski prostor V na sebi ima zbrajanje vektora (kao zbrajanje skalara) $\rightarrow (V,+)$ Abelova i ima množenje vektora skalarom: $1 \cdot v = v$, $(\alpha+\beta)v = \alpha v + \beta v$, $\alpha(v+w) = \alpha v + \alpha w$, $(\alpha\beta)v = \alpha(\beta v)$

Tipičan primjer v.p. je K^n, $n \in \{1,2,\dots\}$
$x = (\alpha_1,\dots,\alpha_n) \in K^n$ $\quad x+y = (\alpha_1+\beta_1,\dots,\alpha_n+\beta_n)$
$y = (\beta_1,\dots,\beta_n) \in K^n$ $\quad \gamma \cdot x = \gamma x = (\gamma\alpha_1,\dots,\gamma\alpha_n)$

Na v.p. najvažnija je LINEARNA KOMBIN. vektora: $\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$ $\quad \alpha_i \in K,\ v_i \in V$

Vektori $v_1, v_2, \dots, v_n$ su LINEARNO ZAVISNI ako možemo naći ne sve jednake 0 skalare $\alpha_1,\dots,\alpha_n$ t.d. $\sum_{i=1}^{n} \alpha_i v_i = 0$. U suprotnom su LINEARNO NEZAVISNI. $(\sum \alpha_i v_i = 0 \Rightarrow \alpha_i = 0\ \forall i)$

Nap. Ako je jedan od tih vektora 0-nulvektor, onda su lin. zavisni.
Npr. $v_1 = 0$ tada možemo pisati $1 \cdot v_1 + 0 \cdot v_2 + \dots + 0 v_n = 0$ $\quad \alpha_1 = 1 \neq 0$

Npr. $V = K^2$ $\quad v_1 = (1,0)$ $\quad v_2 = (0,1)$ $\quad v_3 = (1,1)$
$v_1 + v_2 = v_3 \Rightarrow v_1 + v_2 - v_3 = 0$ $\quad \alpha_1 = \alpha_2 = 1$ $\quad \alpha_3 = -1$ $\quad 1 \neq 0$ $\ -1 \neq 0 \Rightarrow \{v_1, v_2, v_3\}$ lin. zav.

Npr. $V = K^n$ $\quad e_1 = (1,0,\dots,0)$ $\ e_2 = (0,1,0,\dots,0)$ $\ \dots$ $\ e_n = (0,\dots,0,1)$
$$\sum_{i=1}^{n} \alpha_i e_i = 0 = (0,\dots,0) = (\alpha_1,0,\dots,0) + (0,\alpha_2,0,\dots,0) + \dots + (0,\dots,0,\alpha_n) \Leftrightarrow \alpha_1 = \alpha_2 = \dots = \alpha_n = 0 \Rightarrow \text{lin. nez.}$$

Neka $v_1,\dots,v_n \in V$. LINEARNA LJUSKA vektora $v_1,\dots,v_n$, u oznaci $\langle v_1,\dots,v_n \rangle$ $[\{v_1,\dots,v_n\}]$ je skup svih linearnih kombinacija vektora $v_1,\dots,v_n$, tj. $\langle v_1,\dots,v_n \rangle = \{ \sum_{i=1}^{n} \alpha_i v_i\ ;\ \alpha_i \in K \}$.

VAŽNO: $\langle v_1,\dots,v_n \rangle$ je v.p., uz naslijeđene operacije iz V.
Kažemo da je VEKTORSKI POTPROSTOR, $\langle v_1,\dots,v_n \rangle \leq V$

Kako provjeravamo je li $W \leq V$? Ovako: $\forall w_1, w_2 \in W$ $\ \forall \gamma_1, \gamma_2 \in K \overset{?}{\Rightarrow} \gamma_1 w_1 + \gamma_2 w_2 \in W$

Primjer: Pokazujemo $W = \langle v_1,\dots,v_n \rangle \leq V$.
Uzmimo $w_1, w_2 \in W$, $\gamma_1, \gamma_2 \in K$.
$w_1, w_2 \in W \Rightarrow w_1 = \alpha_1 v_1 + \dots + \alpha_n v_n \ \wedge\ w_2 = \beta_1 v_1 + \dots + \beta_n v_n$
Računamo $\gamma_1 w_1 + \gamma_2 w_2 = \gamma_1 (\sum_{i=1}^{n} \alpha_i v_i) + \gamma_2 (\sum_{i=1}^{n} \beta_i v_i) = \overbrace{\sum_{i=1}^{n} \underbrace{(\gamma_1 \alpha_i + \gamma_2 \beta_i)}_{\text{skalar}} v_i}^{\text{lin. komb. vektora } v_1,\dots,v_n} \in W \Rightarrow \boxed{W \leq V}$

DZ. Dokažite da je $\{(x,y,z);\ x-2y+z=0\} \leq K^3 = \{(x,y,z);\ x,y,z \in K\}$.
$S = \{(x,y,z);\ x-2y+z=0\} = \{(x,y,z);\ z = -x+2y\} = \{(x,y,-x+2y);\ x,y \in K\}$
$(x,y,-x+2y) = x(1,0,-1) + y(0,1,2) \Rightarrow S = \{x(1,0,-1) + y(0,1,2),\ x,y \in K\} \Rightarrow S = \langle (1,0,-1), (0,1,2) \rangle$
ovo je upravo definicija lin. ljuske vekt. $(1,0,-1)$, $(0,1,2)$
$\Rightarrow S \leq K^3$

Def. V.p. V je KONAČNODIMENZIONALAN ako postoje $v_1,\dots,v_e \in V$ t.d. $\langle v_1,\dots,v_e \rangle = V$ (tj. svaki vektor iz V je lin. komb. vektora $v_1,\dots,v_n$)

$V = \{0_V\}$ "NUL-PROST."
$v_1 = v_2 = \dots = v_e = 0_V$
kon. dim $\quad \dim\{0_V\} = 0$

$\{0\} \neq V = \langle v_1,\dots,v_e \rangle$ izbacimo sve $v_i = 0_V$, tj. pretp. da $v_i \neq 0$, $\forall i$. Među njima izaberemo najmanji broj n t.d. $e_1,\dots,e_n \in \{v_1,\dots,v_e\}$ t.d. $\langle e_1,\dots,e_n \rangle = V$. Tada $\underbrace{\{e_1,\dots,e_n\}}_{\text{BAZA}}$ lin. nez, a broj n ne ovisi o izboru tih e_i, $\dim V = n$.

Uređena baza: $e = (e_1, e_2, \dots, e_n)$ ili $e' = (e_2, e_1, e_3, \dots, e_n)$ $\quad$ Odsada: baza = uređena baza
Izbor baze omogućuje koordinatizaciju v.p. (DZ - 1.3.) $\quad$ Tada: $\forall v \in V$ $\ \exists! (\alpha_1,\dots,\alpha_n) \in K^n$ t.d. $v = \sum_{i=1}^{n} \alpha_i e_i$

1.3. KOORDINATIZACIJA (od sada svi v.p. kon. dim.)

DZ pročitat 1.3. (nismo niš pisali iz 1.3.)

$e = (e_1, \dots, e_n)$ uređena baza za V $\Rightarrow$ $\forall v \in V$ $\exists! (\alpha_1, \dots, \alpha_n) \in K$ t.d. $v = \alpha_1 e_1 + \dots + \alpha_n e_n$

To nam omogućuje identifikacije:

$$V \leftrightarrow M_{n1}(K), \quad v \longleftrightarrow \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} \quad \text{ili} \quad V \longleftrightarrow M_{1n}(K), \quad v \leftrightarrow [\alpha_1 \ \alpha_2 \ \dots \ \alpha_n]$$

$M_{mn}(K)$ je v.p. svih matrica koje imaju m redaka i n stupaca:

$$A = [\alpha_{ij}] = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n} \\ \vdots & \vdots & & \vdots \\ \alpha_{m1} & \alpha_{m2} & \dots & \alpha_{mn} \end{bmatrix}, \ \alpha_{ij} \in K \qquad \dim M_{mn}(K) = m \cdot n$$

1.4., 1.5., 1.6. LINEARNI OPERATORI

Def. Neka su V, W v.p. Preslikavanje (funkcija) $A: V \to W$ naziva se **LINEARAN OPERATOR** ako vrijedi $A(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 A v_1 + \alpha_2 A v_2$, $\forall \alpha_1, \alpha_2 \in K$, $\forall v_1, v_2 \in V$.

- $\mathcal{L}(V,W)$ – skup svih lin. op iz V u W
  - zbrajamo ih (lin. operatore) "po točkama" $(A+B)v := Av + Bv$ $\quad \forall v \in V$
  - množimo skalarom $(\lambda A) \cdot v := \lambda A v$ $\quad \forall v \in V \ \forall \lambda \in K$

  $\longrightarrow$ $\mathcal{L}(V,W)$ je v.p.

$(A+B)(\alpha_1 v_1 + \alpha_2 v_2) =$
$= A(\alpha_1 v_1 + \alpha_2 v_2) + B(\alpha_1 v_1 + \alpha_2 v_2)$
$= \alpha_1 A v_1 + \alpha_2 A v_2 + \alpha_1 B v_1 + \alpha_2 B v_2$
$= \alpha_1 (A+B) v_1 + \alpha_2 (A+B) v_2$

$(\lambda A)(\alpha_1 v_1 + \alpha_2 v_2) = \lambda A(\alpha_1 v_1 + \alpha_2 v_2) = \lambda(\alpha_1 A v_1 + \alpha_2 A v_2) = \alpha_1 \lambda A v_1 + \alpha_2 \lambda A v_2$

Njegov eksplicitni opis dobijemo u matričnoj formi koristeći izbor baza: $e = (e_1, \dots, e_n)$ za V, $f = (f_1, \dots, f_m)$ za W

Za $A \in \mathcal{L}(V,W)$ napišemo $A e_1 = \alpha_{11} f_1 + \alpha_{21} f_2 + \dots + \alpha_{m1} f_m$
$\vdots$
$A e_n = \alpha_{1n} f_1 + \alpha_{2n} f_2 + \dots + \alpha_{mn} f_m$

$A \Leftrightarrow$ matrica

te na taj način operatoru A pridružujemo matricu

"matrica operatora A u paru baza f i e"

$$A(f,e) = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n} \\ \vdots & \vdots & & \vdots \\ \alpha_{m1} & \alpha_{m2} & \dots & \alpha_{mn} \end{bmatrix}$$
$\quad \uparrow \quad \uparrow \quad \quad \uparrow$
$\quad Ae_1 \ Ae_2 \quad Ae_n$

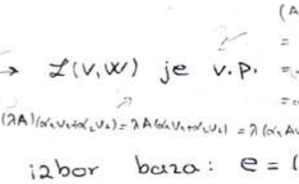
m redova, n stupaca $\quad M_{mn}(K)$

ova identifikacija ovisi o izboru baze e za V i f za W

$\forall$ izborom baza $\mathcal{L}(V,W)$ identificira se s $M_{mn}(K)$,
$\hookrightarrow$ v.p. uz uobičajeno zbrajanje i množenje ~~matrica~~ skalarom

Posebno, $\dim \mathcal{L}(V,W) = \dim M_{mn}(K) = m \cdot n = \dim W \cdot \dim V$

1.7. JEZGRA I SLIKA OPERATORA



$\operatorname{Ker} A \leq V$, $\dim(\operatorname{Ker} A) =: d(A)$

$\operatorname{Im} A \leq W$, $\dim(\operatorname{Im} A) =: r(A)$

Neka je $A \in \mathcal{L}(V,W)$. Tada definiramo:

$\alpha A v + \beta A w = A(\alpha v + \beta w) \in \operatorname{Im} A$

- SLIKA OPERATORA A $\quad R(A) := \operatorname{Im} A := \{Av;\ v \in V\} \leq W$ (DZ) $\quad r(A) := \dim \operatorname{Im}(A)$ $\quad$ RANG
- JEZGRA OPERATORA A $\quad N(A) := \operatorname{Ker} A := \{v \in V;\ Av = 0_V\} \leq V$ (DZ) $\quad d(A) := \dim \operatorname{Ker}(A)$ $\quad$ DEFEKT

$A(\alpha v_1 + \beta w_1) = \alpha A v_1 + \beta A w_1 = 0 \Rightarrow \alpha v_1 + \beta w_1 \in \operatorname{Ker} A$

TM. O RANGU I DEFEKTU

Tm $\boxed{d(A) + r(A) = \dim V}$

Tm. A injektivan $\Leftrightarrow$ $\operatorname{Ker}(A) = 0$ $\Leftrightarrow$ $d(A) = 0$

Tm A surjektivan $\overset{\text{def.}}{\Leftrightarrow}$ $\operatorname{Im}(A) = W$ $\Leftrightarrow$ $r(A) = \dim W$

Nap. Surj. op. može postojati samo ako $\dim W \leq \dim V$.
Jer ako $\dim V < \dim W$, onda prema Tm. o rangu i def.
$r(A) \leq \underbrace{r(A) + d(A)}_{d(A) \geq 0} = \dim V < \dim W \Rightarrow r(A) < \dim W$
$\Rightarrow A$ nije surj.

1.8. DUALAN PROSTOR

ove trebam da bih uopće imala lin.op.

LINEARNI FUNKCIONAL na V je preslikavanje $f: V \to K$, koje zadov. $\boxed{f(\alpha v+\beta w)=\alpha f v+\beta f w}$

K je poseban slučaj v.p. K^n (za $n=1$). $\dim K^n = n \Rightarrow \dim K = 1$

Skup svih lin. funkc. sa V u K nazivamo **dualni prostor od V**, u oznaci $V^*, = V' = \mathcal{L}(V,K)$

Znamo $\dim V^* = \dim \mathcal{L}(V,K) = \dim V \cdot \dim K = \dim V \cdot 1 = \dim V$

$V = K^n = \{(x_1,\dots,x_n);\ x_i \in K,\ i=1,\dots,n\}$

Kako izgledaju lin. funkc. na K^n?

$f: V \to K$ jedan takav lin. funkc.

$(e_1,\dots,e_n)$ baza za K^n

i-to mjesto

$e_i = (0,\dots,0,1,0,\dots,0)$

$\forall x = (x_1,\dots,x_n) \in K^n \quad x = \underbrace{\sum_{i=1}^n \underbrace{x_i}_{\text{skalari}} e_i}_{\text{lin. komb.}} \Rightarrow f(x) = f\left(\sum_{i=1}^n x_i e_i\right) \overset{f\ \text{l.o.}}{=} \sum_{i=1}^n x_i \underbrace{f(e_i)}_{\substack{\text{ove skalare}\\ \text{ozn. sa } \alpha_i}} = \sum_{i=1}^n x_i \alpha_i = \sum_{i=1}^n \alpha_i x_i$

Lin. funkc. f na K^n oblika je $f(x) = \sum_{i=1}^n \alpha_i x_i, \quad \forall x = (x_1,\dots,x_n) \in K^n$

Svaki lin. funkc. f na K^n jedinstveno je određen n-torkom brojeva: $(\underbrace{f(e_1)}_{\alpha_1}, \underbrace{f(e_2)}_{\alpha_2}, \dots, \underbrace{f(e_n)}_{\alpha_n})$. (tj. djelovanjem na bazu)

$\Rightarrow$ što znači da V^* identificiramo s K^n izborom baze $(e_1,\dots,e_n)$ te nam to daje novi dokaz za $\dim V^* = n = \dim K^n = \dim V$.

2.4.

- Dualni prostor V^* vekt. prostora V sastoji se od linearnih funkcionala $f: V \to K$ ($K = \mathbb{R}$ ili $\mathbb{C}$)
  $\hookrightarrow f(\alpha v + \beta w) = \alpha f(v) + \beta f(w) \quad \forall \alpha,\beta \in K\ \forall v,w \in V$
- $\dim V^* = \dim V$
- V^* je vektorski prostor uz ovako def.:
  - zbrajanje $(f+g)(v) := fv + gv \quad \forall f,g \in V^*\ \forall v \in V$
  - mn. skal. $(\lambda f)(v) := \lambda(fv) \quad \forall f \in V^*\ \forall \lambda \in K\ \forall v \in V$
- Svaki lin. funkc. $f \in V^*$ je lin.op. $f \in \mathcal{L}(V,K)$ (po def.) te je zato f (kao i svaki lin.op.) određen djelovanjem na bazu $e = (e_1,\dots,e_n)$ v.p. V vrijednostima $f(e_1),\dots,f(e_n)$ (koji su iz K)

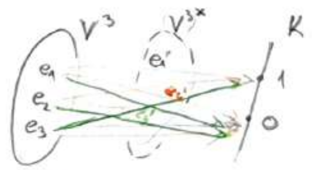
  Ako zadamo $\lambda_1,\dots,\lambda_n \in K$ onda $f(\underbrace{\sum_{i=1}^n \alpha_i e_i}_{\in V}) = \sum_{i=1}^n \alpha_i \underbrace{f(e_i)}_{\lambda_i =} = \sum_{i=1}^n \lambda_i \alpha_i$

Sada primjenimo iskazano za konstrukciju posebnog tipa lin. funkc.
Uzmemo bilo koju bazu $e = (e_1,\dots,e_n)$ za V. Definiramo n-torku funkc. $e' = (e_1',\dots,e_n')$:

$\lambda_1=1$ ost$=0$	$e_1'(e_1)=1$	$e_1'(e_2)=0$	$\dots$	$e_1'(e_n)=0$
$\lambda_2=1$ ost$=0$	$e_2'(e_1)=0$	$e_2'(e_2)=1$	$\dots$	$e_2'(e_n)=0$
	$\vdots$			
$\lambda_n=1$ ost$=0$	$e_n'(e_1)=0$	$e_n'(e_2)=0$	$\dots$	$e_n'(e_n)=1$

$e_1'(1e_1+0e_2+0e_3) = 1e_1'e_1 + 0e_1'e_2 + 0e_1'e_3 = 1$

$= \boxed{e_i'(e_j) = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases}}$



Lema: $e' = (e_1',\dots,e_n')$ je baza prostora V^*, naziva se <u>DUALNA BAZA BAZE e</u>.
Nadalje, za svaki $f \in V^*$ vrijedi prikaz $f = \sum_{i=1}^n f(e_i)e_i'$ u bazi e'.

DOKAZ:

- $\dim V^* = \dim V = n$ pa da bi $e' = (e_1',\dots,e_n')$ bila baza trebamo pokazati samo lin.nez. funkc. $e_1',\dots,e_n'$

  $\lambda_1 e_1' + \lambda_2 e_2' + \dots + \lambda_n e_n' = 0^{\text{nul-funkcional}}$ / djelujemo na bazu $e = (e_1,\dots,e_n)$

  $\to \lambda_1 e_1'(e_i) + \lambda_2 e_2'(e_i) + \dots + \lambda_n e_n'(e_i) = 0(e_i) \to \lambda_i = 0 \quad i=1,2,\dots,n \to \{e_1',e_2',\dots,e_n'\}$ lin.nez $\to \boxed{e' \text{ baza od } V^*}$ //

- $g := \sum_{i=1}^n f(e_i)e_i' \leftarrow$ ovako izgleda zapis lin. funkc. koji je određen djelovanjem na bazi

  $g(e_j) = \sum_{i=1}^n f(e_i)\underbrace{e_i'(e_j)}_{=1 \text{ za } i=j} = f(e_j) \Rightarrow f$ i g se podudaraju na bazi $\to f = g \to \boxed{f = \sum_{i=1}^n f(e_i)e_i'}$ //

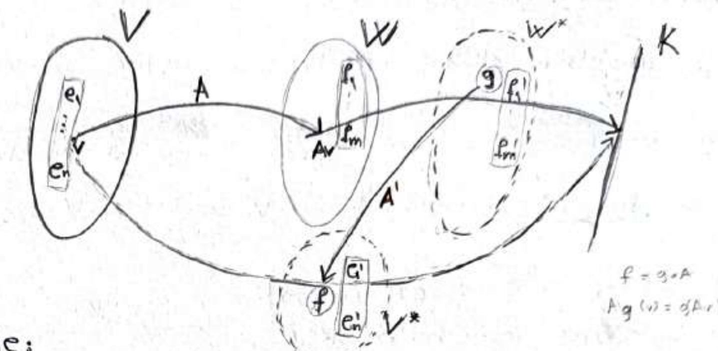
1.10. ADJUNGIRANI OPERATOR

V, W v.p. $\quad \boxed{A \in \mathcal{L}(V,W)} \xrightarrow{\text{def. lin. op.}}$ to znači A preslikavanje $A: V \to W$ koje $A(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 A v_1 + \alpha_2 A v_2 \quad \forall \alpha_1, \alpha_2 \in K \ \forall v_1, v_2 \in V$

$g \in W^* = \mathcal{L}(W, K)$

$$\begin{tikzcd} V \arrow[r, "A"] \arrow[rr, bend left, dashed, "f \in V^*"] & W \arrow[r, "g \in W^*"] & K \end{tikzcd}$$

kompozicija preslikavanja



$f = g \circ A$

$A'g(v) = g(Av)$

Ali kakvo preslikavanje smo dobili?

$f: V \to K \qquad \underset{\in V^*}{f}(v) := \underset{\in W^*}{g}(Av) \qquad \forall v \in V$

Utvrdimo da je to linearno preslikavanje:

$$f(\alpha_1 v_1 + \alpha_2 v_2) \overset{\text{def}}{=} g(A(\alpha_1 v_1 + \alpha_2 v_2)) \overset{A \text{ lin. op}}{=} g(\alpha_1 \overset{\in W}{A v_1} + \alpha_2 \overset{\in W}{A v_2}) \overset{g \text{ lin. funkc.}}{=} \alpha_1 \underbrace{g A v_1}_{f v_1 \in K} + \alpha_2 \underbrace{g A v_2}_{f v_2 \in K} \overset{\text{def}}{=} \alpha_1 f v_1 + \alpha_2 f v_2$$

$\to f$ lin. preslik. $\Rightarrow f \in V^*$

Konstruirali smo preslikavanje: $\left.\begin{array}{l} W^* \longrightarrow V^* \\ g \longmapsto f = g \circ A =: gA \end{array}\right\}$ ovo preslikavanje označavamo A'

A' zadovoljava sljedeće: $A'g = f = g \circ A \overset{\text{pišemo}}{=} gA$ ili raspisano $\underbrace{(A'g)}_{f}(v) = g(Av) \quad \forall v \in V \ \forall g \in W^*$

Prop. **$A' \in \mathcal{L}(W^*, V^*)$, a nazivamo ga <u>ADJUNGIRANI OPERATOR</u> OD $A \in \mathcal{L}(V,W)$.**

DOKAZ:

$$A'(\overbrace{\beta_1 g_1 + \beta_2 g_2}^{\in W^*}) \overset{?}{=} \beta_1 A' g_1 + \beta_2 A' g_2 \qquad \forall \beta_1, \beta_2 \in K \quad \forall g \in W'$$

$$A'(\beta_1 g_1 + \beta_2 g_2)(v) \overset{?}{=} \beta_1 (A'g_1)(v) + \beta_2 (A'g_2)(v)$$

$$\underbrace{(\beta_1 g_1 + \beta_2 g_2)}_{\text{l.o.}} A(v) \overset{?}{=} \beta_1 \underbrace{(A'g_1)(v)}_{g_1 A(v)} + \beta_2 \underbrace{(A'g_2)(v)}_{g_2 A(v)} \ \checkmark \ \to \boxed{A' \in \mathcal{L}(W^*, V^*)}$$

Def. $A': W^* \to V^*$
$A'(g) = f$
A' ADJ. OP.

To je <u>KANONSKA KONSTRUKCIJA ADJUNGIRANOG OPERATORA A'</u>. Ne ovisi o izboru baza. Izborom baza (e) i (f) za V i W možemo konstruirati odgovarajuće dualne baze (e') i (f').

Prop. 1.4. **U paru baza (e', f') matrica lin. op. A' jednaka je transponiranoj matrici lin. op. A u paru baza (f, e), tj. $A'_{(e', f')} = A_{(f,e)}$.**

usm. kol.

DZ DOKAZ

DOKAZ:

Neka su $A_{(f,e)}$ i $A'_{(e',f')}$ zadane na sljedeći način:

$$A_{(f,e)} = \begin{array}{c} \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & & \alpha_{2n} \\ \vdots & \vdots & & \vdots \\ \alpha_{m1} & \alpha_{m2} & & \alpha_{mn} \end{bmatrix} \\ \begin{array}{cccc} \uparrow & \uparrow & & \uparrow \\ Ae_1 & Ae_2 & & Ae_n \end{array} \end{array} \qquad A'_{(e',f')} = \begin{array}{c} \begin{bmatrix} \beta_{11} & \beta_{12} & \cdots & \beta_{1m} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2m} \\ \vdots & \vdots & \beta_{ij} & \vdots \\ \beta_{n1} & \beta_{n2} & \cdots & \beta_{nm} \end{bmatrix} \\ \begin{array}{cccc} \uparrow & \uparrow & & \uparrow \\ A'f_1' & A'f_2' & & A'f_m' \end{array} \end{array}$$

Ekvivalentno njima je:

$$\begin{cases} Ae_1 = \alpha_{11} f_1 + \alpha_{21} f_2 + \ldots + \alpha_{m1} f_m \\ Ae_2 = \alpha_{12} f_1 + \alpha_{22} f_2 + \ldots + \alpha_{m2} f_m \\ \quad \vdots \\ Ae_n = \alpha_{1n} f_1 + \alpha_{2n} f_2 + \ldots + \alpha_{mn} f_m \end{cases} \qquad \begin{cases} A'f_1' = \beta_{11} e_1' + \beta_{21} e_2' + \ldots + \beta_{n1} e_n' \\ A'f_2' = \beta_{12} e_1' + \beta_{22} e_2' + \ldots + \beta_{n2} e_n' \\ \quad \vdots \\ A'f_m' = \beta_{1m} e_1' + \beta_{2m} e_2' + \ldots + \beta_{nm} e_n' \end{cases}$$

Raspišimo β_{ij} (i-ta koordinata slike j-tog vektora baze f' u bazi e'):

$$\beta_{ij} = \beta_{ij} \cdot e_i'(e_i) = \Big(\sum_{k=1}^{n} \beta_{kj} e_k'\Big)(e_i) = (A'f_j')(e_i) = f_j'(Ae_i)$$

$$= f_j'\Big(\sum_{k=1}^{m} \alpha_{ki} f_k\Big) = \sum_{k=1}^{m} \alpha_{ki} f_j'(f_k) = \alpha_{ji} \quad \rightsquigarrow \quad \boxed{A_{(f,e)} = A'_{(e',f')}}$$

1.11. KANONSKI IZOMORFIZAM

def. → lin. op koji je bijekcija

$$V \longrightarrow V^* \longrightarrow (V^*)^* =: V^{**}$$

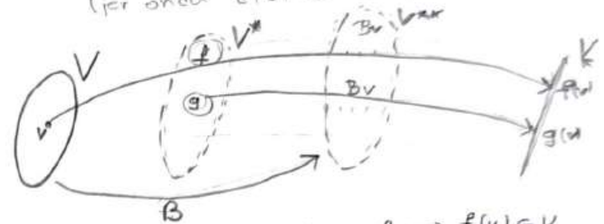
B: postoji kanonski izomorfizam, tj. kanon. lin. op. koji je bijekcija

Znamo: $\dim V^{**} = \dim V^* = \dim V$

$\Rightarrow \dim V^{**} = \dim V$

Tm o rangu i defektu → treba konstruirati kanonski lin. op. koji je injekcija

(jer onda $d(B)=0 \to r(B)=\dim V = \dim V^{**}$)



Tražimo $B \in \mathcal{L}(V, V^{**})$ koji je injekcija.

$f \in V^*$ $v \in V$ → $f(v) \in K$

→ $v \mapsto f(v)$ v varira po V dobimo f

Međutim, možemo fiksirati $v \in V$ i promatrati preslikavanje $V^* \to K$, $f \mapsto f(v) \in K$.

To preslikavanje nazovemo "Bv", $(Bv)(f) = f(v)$, $\forall f \in V^*$ $B(v) = $ "Bv"

$V^* \to K$ — ovo varira po elementima iz V^*

$B(v)$ "Bv"

Lema. $\forall v \in V$ je "Bv" lin. funkc. na V^*, tj. $Bv \in V^{**}$.

DOKAZ

Neka su $f, g \in V^*$, $\alpha, \beta \in K$

$(Bv)(\alpha f + \beta g) \overset{?}{=} \alpha (Bv)(f) + \beta (Bv)(g)$ $=\mathcal{L}(V^*, K)$

$\underbrace{(\alpha f + \beta g)}_{l.o.}(v) \overset{?}{=} \alpha f(v) + \beta g(v)$ ✓ $\Rightarrow \boxed{Bv \in V^{**}}$

Lema $B : V \to V^{**}$ je lin. op. $(B(v) \mapsto Bv)$

DOKAZ

Neka su $v_1, v_2 \in V$, $\alpha_1, \alpha_2 \in K$

jer kad B djeluje na neki V, dobimo neki V^{**}, koji onda mora djelovati na neki $f \in V^*$ ← $\underset{\in V^{**}}{B(\alpha_1 v_1 + \alpha_2 v_2)}(f) \overset{?}{=} \alpha_1 B v_1(f) + \alpha_2 B v_2(f)$

"$B(\alpha_1 v_1 + \alpha_2 v_2)$"

$f(\alpha_1 v_1 + \alpha_2 v_2) \overset{?}{=} \alpha_1 f(v_1) + \alpha_2 f(v_2)$ ✓ $f \in V^*$ $\Rightarrow \boxed{B \text{ je lin.op.}}$

Lema B je injekcija, tj. traženo kanonsko preslikavanje

DOKAZ:

$\mathrm{Ker}(B) \overset{?}{=} \{0\}$, tj. $Bv = 0 \leftrightarrow v = 0$

Pretp. $Bv(f) = 0(f) \in V^{**} \to f(v) = f(0) = 0 \in K$ $\forall f \in V^* \xrightarrow{?} v = 0$

Ako $v \neq 0$ nadop. do baze $e = (e_1, \dots, e_n)$ za V u kojoj je $e_1 = v$. Samo uzmemo $f = e_1'$. $0 = f(v) = e_1'(e_1) = 1$ ↯ $\to v = 0 \to \mathrm{Ker}(B) = \{0\} \to \boxed{B \text{ inj.}}$

Rezimirajmo:

skup svih V **KANONSKI IZOMORFIZAM**

$B : V \to V^{**}$

$(Bv)(f) = f(v)$ $\forall f \in V^*$ $\forall v \in V$

$Bv : V^* \to K$

Neka je također $C : W \to W^{**}$ kan. izom.:

$(Cw)(g) = g(w)$ $\forall g \in W^*$ $\forall w \in W$

$$\begin{array}{ccc} V & \xrightarrow{B} & V^{**} \\ A \downarrow & & \downarrow A'' \\ W & \xrightarrow{C} & W^{**} \end{array}$$

B, C bijekcije

diagram: $A''B = CA$

$A'' = CAB^{-1}$

adj. op. od A

$A' \in \mathcal{L}(W^*, V^*)$

↓ njegov adjungirani

$(A')' \in \mathcal{L}(V^{**}, W^{**})$

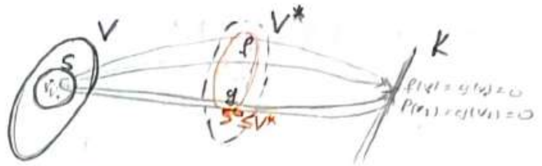
Tm. $\boxed{(A')' = A''}$

BEZ DOKAZA

(tj. ako V i V^{**} identificiramo putem B, W i W^{**} identific. putem C, onda $(A')' = A \equiv A'$)

identifikacija

1.12. ANIHILATORI



$S \subseteq V$ (podskup)

$S^\circ := \{ f \in V^* ;\ f(s) = 0,\ \forall s \in S \}$ $\quad S^\circ$ - "ANIHILATOR OD S"

Npr. $S \neq 0 \Rightarrow \emptyset = V'$ (po def.)

Lema $\boxed{S^\circ \text{ je v.p.} \leq V^*}$

DOKAZ:

DZ

Treba pokazati: $\forall f, g \in S^\circ\ \forall \alpha, \beta \in K \to \alpha f + \beta g \in S^\circ$

$\alpha f(s) + \beta g(s) = \alpha \cdot 0 + \beta \cdot 0 = 0 \to \alpha f + \beta g \in S^\circ \to \boxed{S^\circ \text{ v.p.}}$ $\quad S^\circ \leq V^*$

$T \subseteq V^*$

$T^\circ = \{ v \in V ;\ f(v) = 0,\ \forall f \in T \}$

Lema $\boxed{T^\circ \text{ je vek. ptp. u } V}$

DOKAZ:

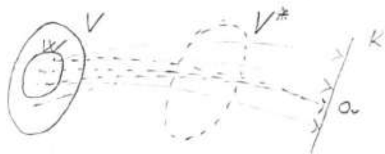
DZ

$v, w \in T^\circ \quad \alpha, \beta \in K \quad \alpha v + \beta w \overset{?}{\in} T^\circ$

$f(\alpha v + \beta w) \overset{f \in T \to f \in V^*}{=} \alpha \underbrace{f(v)}_{0} + \beta \underbrace{f(w)}_{0} = 0 \to \boxed{T^\circ \leq V}$

Lema VAŽNO!

Neka je $W \leq V$. Tada:
1) $\dim W^\circ = \dim V - \dim W$
2) $(W^\circ)^\circ = W$



$(\overbrace{e_1, \dots, e_k}^{W}, e_{k+1}, \dots, e_n)$ V

$(e_1', \dots, e_k', \underbrace{e_{k+1}', \dots, e_n'}_{\text{baza za } W^\circ})$ dualna

DOKAZ:

DZ

1) Neka je $e = (e_1, \dots, e_n)$ baza za V t.d. je $(e_1, \dots, e_k)$ baza za W (ukoliko $W \neq \emptyset$).
Neka je $e' = (e_1', \dots, e_n')$ dualna baza baze e.
Tada $e_{k+1}', \dots, e_n' \in W^\circ$ (jer $e_{k+1}'(e_i) = 0 \quad \forall i = 1, \dots, k$)
$f \in W^\circ \to f = f(e_1)e_1' + \dots + f(e_n)e_n' = f(e_{k+1})e_{k+1}' + \dots + f(e_n)e_n' \to (e_{k+1}', \dots, e_n')$ baza za W°
Dakle $\dim W^\circ = n - (k+1) + 1 = n - k = \boxed{\dim V - \dim W = \dim W^\circ}$

2) $(\underbrace{e_{k+1}', \dots, e_n'}_{\text{baza za } W^\circ}, e_1', \dots, e_k')$ baza za V'

dualna $(e_{k+1}, \dots, e_n, \underbrace{e_1, \dots, e_k}_{\text{baza za } (W^\circ)^\circ \ =\ \text{baza za } W})$

1.13. RANG ADJUNGIRANOG OPERATORA

- to je isto kao i činjenica da su matrica i njoj transponirana matrica istog ranga

Tm. 1.8. Neka je $A \in \mathcal{L}(V, W)$. Tada:
i) $\mathrm{Ker}(A') = (\mathrm{Im}(A))^\circ \quad (\Leftrightarrow \mathrm{Im}\,A = (\mathrm{Ker}\,A')^\circ)$
ii) $\mathrm{Ker}(A) = (\mathrm{Im}(A'))^\circ \quad (\Leftrightarrow \mathrm{Im}\,A' = (\mathrm{Ker}\,A)^\circ)$
iii) $r(A') = r(A)$

+ prepisati dokaz

2. MINIMALNI POLINOM LINEARNOG OPERATORA

V v.p. nad $K = \mathbb{R}$ ili $\mathbb{C}$

$\mathcal{L}(V) := \mathcal{L}(V,V)$ skup svih lin. op. $A: V \to V$

$\mathcal{L}(V)$ zadovoljava:

① $A, B \in \mathcal{L}(V),\ \lambda, \mu \in K \Rightarrow \lambda A + \mu B \in \mathcal{L}(V)$

$(\lambda A + \mu B)(v) = \lambda Av + \mu Bv \quad \forall v \in V$

$\mathcal{L}(V)$ je v.p., od prije, iz koordinatizacije znamo $\dim \mathcal{L}(V) = n^2$, $n = \dim V$.

② $A, B \in \mathcal{L}(V) \Rightarrow AB \in \mathcal{L}(V)$

$(AB)(v) := A(Bv) \quad \forall v \in V$

Ta operacija (množenje operatora) je jako dobra jer zadovoljava:

a) asocijativnost $(AB)C = A(BC)$

b) što omogućuje definiciju potencija A^k, $k \geq 1$, koje su definirane induktivno:
$A^1 = A$, $A^k = A^{k-1} \cdot A \quad \forall k \geq 2$

c) kompatibilna s množenjem skalarom: $(\lambda A)B = A(\lambda B) = \lambda(AB) \quad \forall \lambda \in K\ \forall A, B \in \mathcal{L}(V)$

d) kompatibilna sa strukturom v.p. na $\mathcal{L}(V)$, tj. ①: $(\lambda A + \mu B)C = \lambda AC + \mu BC$
$C(\lambda A + \mu B) = \lambda CA + \mu CB$

e) jedinični operator: $I \in \mathcal{L}(V)$, $I = I_V$, $Iv = v \quad \forall v \in V$ $\quad (IA = AI = A \quad \forall A \in \mathcal{L}(V))$

f) nuloperator: $O \in \mathcal{L}(V) \quad O_v = O(v) \quad \forall v \in V$

① ∧ ② → $(\mathcal{L}(V), +, \cdot)$ algebra (asocijativna algebra s 1)

③ $A \in \mathcal{L}(V)$ je REGULARAN / INVERTIBILAN ako postoji $B \in \mathcal{L}(V)$ t.d. $AB = BA = I$.
Ako takav B postoji, onda je i jedinstven, u oznaci $A^{-1} = B$.
VAŽNO! Tm. o rangu i defektu $\overset{\text{slijedi}}{\underset{\text{za } A \in \mathcal{L}(V)}{\Rightarrow}}$ Ekvivalentne su tvrdnje: A reg, A bij., A inj. ($\mathrm{Ker} A = \{0\}$) A surj. ($\mathrm{Im} A = V$)

④ (KOORDINATIZACIJA) Neka je $e = (e_1, \dots, e_n)$ uređena baza za V

$A \in \mathcal{L}(V) \longmapsto A_{(e,e)} =: A_{(e)} \in M_{n \times n}(K)$ ← ovo je bilo (DZ) (str.2.)

$= \mathcal{L}(V,V)$

t.d. $(\lambda A + \mu B)(e) = \lambda Ae + \mu Be$ (suma matrica) $\quad \forall \lambda, \mu \in K \quad \forall A, B \in \mathcal{L}(V)$

$(AB)(e) = Ae \cdot Be$ (produkt matrica)

$(\lambda A + \mu B)_{(e)} = \lambda A_{(e)} + \mu B_{(e)}$
$(AB)_{(e)} = A_{(e)} \cdot B_{(e)}$
Piše li zapravo to?

$IV_{(e)} = I_{(e)} = I_n = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$

$O_{(e)} = [0]$

Preslikavanje $\mathcal{L}(V) \to M_{n \times n}(K)$ je bijekcija te izborom baze možemo raditi eksplicitne račune u $M_{n \times n}(K)$ i onda se vratiti nazad.

⑤ Primjena od ④

$\det A := \det A_{(e)}$ $\quad$ A - lin. op., $A_{(e)}$ - matrica

Problem u ovoj definici može nastupiti ako bi izborom neke druge baze $e' = (e_1', \dots, e_n')$ dobili $\det A \neq \det A_{(e)}$. No, uvijek je $\det A_{(e)} = \det A_{(e')}$, zbog matrice prijelaza (DZ)

$$A_{(e')} = T^{-1} A_{(e)} T \quad / \det$$

$$\det A_{(e')} = \underbrace{\det T^{-1}}_{\in \mathbb{R}} \underbrace{\det A_{(e)}}_{\in K} \underbrace{\det T}_{\in \mathbb{R}}$$

$$\det A_{(e')} = \det A_{(e)}$$

→ $\det A$ ne ovisi o izboru baze

"polinom u varijabli x"

(6.) $p(x) = a_0 + a_1 x + \dots + a_n x^n$ polinom s koef. $a_0, \dots, a_n \in K = \mathbb{R}$ ili $\mathbb{C}$

Operacije (1) i (2) omogućuju definiciju:

$\forall A \in \mathcal{L}(A) \quad \underbrace{p(A) = a_0 I + a_1 A + \dots + a_n A^n}_{\text{lin. komb. operatora } I, A, \dots, A^n} \in \mathcal{L}(A)$ "evaluacija polinoma u operatoru"

Npr. $p(x) = 1 + 2x + x^2 \rightarrow p(A) = I + 2A + A^2$

Nap. U ovom kolegiju, kao i u drugim algebarskim kolegijima je $A^0 = I \quad \forall A \in \mathcal{L}(V)$, pa onda posebno, $0^0 = I$.

Opća definicija kaže $p(A) = a_0 A^0 + a_1 A + \dots + a_n A^n$ → def. je dobra jer po tm o jednakosti polinoma (iz uvoda) znamo da je polinom jedinstveno određen svojim koef. i stupnjem

(7.) Eksplicitnim formulama za zbroj i umnožak polinoma nalazimo:

$p, g \in K[x]$, $r := pg \rightarrow \forall A \in \mathcal{L}(V)$: a) $(\lambda p + \beta g)(A) = \lambda p(A) + \beta g(A)$

b) $r(A) = pg(A) = p(A) g(A)$

To omogućuje da se identitet iz "svijeta polinoma" prebaci u "svijet l.o."

Npr. $(x+1)^2 = x^2 + 2x + 1 \Rightarrow (A+I)^2 = A^2 + 2A + I$

$(x^2 - 1) = (x-1)(x+1) \Rightarrow (A^2 - I) = (A - I)(A + I)$

DZ Kako to dokazati iz a) i b) ?!

$p(x) = x+1 \quad g(x) = x+1 \quad (p \cdot g)(x) = (x+1)^2$

$p(A) = A+I \quad g(A) = A+I \quad (pg)(A) = (A+I)^2$

(8.) Poseban polinom vezan za $A \in \mathcal{L}(V)$ je njeov "KARAKTERISTIČAN POLINOM" $k_A(x) := \det(xI - A)$ OPREZ: Iz LA znamo $\det(A - xI)$, ali $\stackrel{!}{=} (-1)^n k_A(x)$

↳ normiran (vodeći koef. = 1)

Fix $A \in \mathcal{L}(V)$ $\quad k_A(x) = 1x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 \quad a_0, \dots, a_n \in K$

Tm. HAMILTON-CAYLEYJEV TM.

$$k_A(A) = A^n + a_{n-1} A^{n-1} + \dots + a_1 A + a_0 I = \underbrace{0}_{\text{nul. op.}}$$

Kažemo: A poništava svoj karakteristični polinom.

Linearna kombinacija linearnih operatora $I, A, A^2, \dots, A^n$ jednaka 0_{op}.

Ta je lin. kombinacija netrivijalna jer je koef. uz A^n jednak 1.

CILJ: Naći najmanji $m \geq 1$ t.d. $\exists \mu_0, \mu_1, \dots, \mu_{m-1} \in K$ t.d. $A^m - \mu_{m-1} A^{m-1} - \dots - \mu_1 A - \mu_0 I = 0_{op}$

Otuda slijedi definicija minimalnog polinoma.

$A^m = \mu_{m-1} A^{m-1} + \dots + \mu_1 A + \mu_0 I$ ← lin. komb. lin. op. $I, \dots, A^{m-1}$

Konstrukcija iz gornjeg:

Izaberemo jedinstveni $m \geq 1$ t.d.:
- a) $I, \dots, A^{m-1}$ su lin. nez.
- b) $I, \dots, A^{m-1}, A^m$ su lin. zav.

} to možemo jer su prema H-C-tmu $I, A, \dots, A^n$ lin. zav. (jer koef. uz $A^n = 1$)

$\exists \lambda_0, \dots, \lambda_m \in K$ ne svi $= 0$ t.d. $\lambda_0 I + \lambda_1 A + \dots + \lambda_m A^m = 0$

Nikako ne može biti $\lambda_m = 0$, jer tada $\lambda_0 I + \lambda_1 A + \dots + \lambda_{m-1} A^{m-1} = 0$,

pa zbog a) bi bilo $\forall \lambda_i = 0 \quad \forall i = 1, \dots, m-1$ ↯ Dakle možemo podijeliti sa λ_m.

$\lambda_0 I + \lambda_1 A + \dots + \lambda_m A^m = 0 \quad / : \lambda_m$

$$\underbrace{\frac{\lambda_0}{\lambda_m}}_{-\mu_0 :=} I + \underbrace{\frac{\lambda_1}{\lambda_m}}_{-\mu_1 :=} A + \dots + \underbrace{\frac{\lambda_{m-1}}{\lambda_m}}_{-\mu_{m-1} :=} A^{m-1} + A^m = 0$$

Def. MINIMALNI POLINOM

$\mu_A(x) := x^m - \mu_{m-1} x^{m-1} - \dots - \mu_1 x - \mu_0$

Normirani polinom najmanjeg stupnja kojeg A poništava.

Prop.
1) $m \leq n = \deg k_A$
2) $\mu_A(A) = 0$

Tm. KARAKTERIZACIJA MINIMALNOG POLINOMA

Neka je $A \in \mathcal{L}(V)$. Tada je:

① μ_A je jedinstven normiran polinom najmanjeg stupnja kojeg A poništava

② Ako A poništava polinom f, onda $\mu_A | f$

③ $\mu_A | k_A$ i $\deg \mu_A = \underline{m \leq n} = \dim V = \dim k_A$

DOKAZ.

① Ukoliko bi A poništavao neki polinom $a_0 + a_1 x + \dots + \overset{\neq 0}{a_r} x^r$ $r < m$, (→ stupanj min. pol.)
imali bismo netrivijalnu lin. komb. $\alpha_0 I + \alpha_1 A + \dots + \overset{\neq 0}{\alpha_r} A^r = 0$ → $I, A, \dots, A^r$ lin. zav. ↯ → m najmanji
(jer su podskup lin. nez. skupa $I, A, \dots, A^r, \dots, A^{m-1}$)
Ako A poništava normirani polinom stupnja m $g = x^m + b_{m-1}x^{m-1} + \dots + b_1 x + b_0$,
tj. $A^m + b_{m-1}A^{m-1} + \dots + b_1 A + b_0 I = 0_{op}$, a min. pol. je oblika $\mu_A = x^m +$ "niže potencije",
$\Rightarrow g - \mu_A$ je nul. op ili $\deg(g - \mu_A) < m$
$(g - \mu_A)(A) = g(A) - \mu_A(A) = 0 - 0 = 0_{op} \xrightarrow{①} \boxed{g = \mu_A}$ (jedinstv.)

(DZ) ② Ako A poništava $f(x)$, onda prema ① $\deg f(x) \geq m$ i $f(x) = h(x)\mu_A(x) + r(x)$ (ostatak st < m)

$\Rightarrow \underbrace{f(A)}_{0} = h(A)\underbrace{\mu_A(A)}_{0} + r(A) \to r(A) = 0 \to r(x) = 0 \to \boxed{\mu_A(x) | f(x)}$

(DZ) ③ Slijedi direktno iz ② i H-C-tma.

Tm. 2.11. 2.13. Neka je $A \in \mathcal{L}(V)$, $\lambda \in K$. Tada vrijedi:
Lin. op. $A - \lambda I$ regularan akko $\mu_A(\lambda) \neq 0$.

regularan ≡ ima inverz

DOKAZ

Svodi se pomakom na slučaj $\lambda = 0$ koji razmatramo.

(ovo ZMAD) ⇐ Dokazujemo: A regularan akko $\mu_A(0) \neq 0$.

(⇒) $\mu_A(x) = x^m - \mu_{m-1}x^{m-1} - \dots - \mu_1 x - \mu_0$

$\mu_A(0) = -\mu_0$

Pretp. A regularan. Ali $\mu_A(0) = 0$ tj. $\mu_0 = 0$ (jer A reg.)

$\mu_A(x) = x^m - \mu_{m-1}x^{m-1} - \dots - \mu_1 x \to 0 = \mu_A(A) = A^m - \mu_{m-1}A^{m-1} - \dots - \mu_1 A \;\; / \cdot A^{-1}$

$\to 0 = 0 \cdot A^{-1} = A^m \cdot A^{-1} - \mu_{m-1}A^{m-1}A^{-1} - \dots - \mu A A^{-1}$

$0 = A^{m-1} - \mu_{m-1}A^{m-2} - \dots - \mu_1 I$ ↯ jer ta relacija pokazuje da su $I, A, A^2, \dots, A^{m-1}$ lin. zav. što nije istina

$\boxed{\mu_A(\lambda) = 0}$

(⇐) Pretp. $\mu_A(0) \neq 0$, tj. $-\mu_0 \neq 0$

$0 = \mu_A(A) = A^m - \mu_{m-1}A^{m-1} - \dots - \mu_1 A - \mu_0 I$

$\mu_0 I = A^m - \mu_{m-1}A^{m-1} - \dots - \mu_1 A \;\; /: \mu_0 \to I = A\underbrace{\left(\frac{1}{\mu_0}A^{m-1} - \frac{\mu_{m-1}}{\mu_0}A^{m-2} - \dots - \frac{\mu_1}{\mu_0}I\right)}_{A^{-1}} \to \boxed{A \text{ regularan}}$

VAŽNO! **Def.** **SPEKTAR OPERATORA** $A \in \mathcal{L}(V)$, ozn. σ_A je skup svih $\lambda \in K$ t.d. $A - \lambda I$ nije regularan, tj. nije invertibilan ($\leftrightarrow$ $A - \lambda I$ nije inv.)

(tj. SPEKTAR OPERATORA $A \in \mathcal{L}(V)$, ozn. σ_A, je skup svih $\lambda \in K$ t.d. $\mu_A(\lambda) = 0$)

HAMILTON-CAYLEY

$\sigma_A = \{\lambda \in K ; k_A(\lambda) = 0\}$ konačan skup, najviše $\underline{n}$ nul.t.
↓ neke se javljaju više puta

prethodni tm. ↗ $= \{\lambda \in K, \mu_A(\lambda) = 0\}$ konačan skup

(preuzima najviše m razl. nultočaka)

$m = \deg \mu_A$

3. NILPOTENTNI OPERATORI

INVARIJANTNI POTPROSTORI (važna stvar u v.p.)

Def. $A \in \mathcal{L}(V)$. Kažemo da je ptp. $W \subseteq V$ A-invarijantan ako $\forall w \in W \Rightarrow Aw \in W$.

znači $A|_W \in \mathcal{L}(W)$

VAŽAN PRIMJER!

- Svaki svojstveni potprostor je invarijantan. Svojst. ptp $\neq \{0_V\}$ nul-ptp → uvijek A-invar. $Ker(A-\lambda I)$

$\lambda \in \sigma_A = \{\mu \in K;\ A-\mu I \text{ nije invert.}\} = \{\mu \in K;\ k_A(\mu)=0\} = \{\mu \in K;\ \mu_A(\mu)=0\}$

Pogledajmo ptp. $W = \underbrace{Ker(A-\lambda I)}_{(V_\lambda = \{v \in V;\ Av = \lambda v\})} = \{v \in V;\ (A-\lambda I)v = 0\}$

nije invert. $\rightarrow \exists v \in V$ t.d. $v \neq 0$ $(A-\lambda I)v = 0 \leftrightarrow$ $Av = \lambda I v = \lambda v \rightarrow \lambda$ sv.vr., $v \neq 0$ sv.v.

Tvrdimo: W je A-invar. ptp.

$w \in W \rightarrow (A-\lambda I)w = 0 \rightarrow Aw = \lambda w \in W \rightarrow W$ je A-invar. ptp.

- Trivijalan primjer: $\{0_V\}$ A-invar. $\forall A \in \mathcal{L}(V)$ (DZ) $A0_V = 0_V \in \{0_V\}$ $\forall A \in \mathcal{L}(V)$

Nap. "Nasumice" uzmemo $\lambda \in K$, opet možemo formirati $Ker(A-\lambda I)$ koji je opet A-invar., ali $\lambda \notin \sigma_A \rightarrow Ker(A-\lambda I) = \{0_V\}$!!!

$A-\lambda I$ je reg. $\rightarrow \mu_A(\lambda) \neq 0$; $= \{v \in V;\ (A-\lambda I)v = 0\} \rightarrow v = 0_V$

SUMA POTPROSTORA $W_1, \dots, W_k \subseteq V$

$$\underbrace{W_1 + W_2 + \dots + W_k}_{\sum_{i=1}^k W_i} \overset{def.}{=} \left\{ \sum_{i=1}^k w_i \ ;\ w_i \in W_i \right\}$$

$W_1 + W_2 \leq V$

Lema $\sum_{i=1}^k W_i \leq V$

DOKAZ:

Treba pokazati: $x, y \in \sum_{i=1}^k W_i$ i $\alpha, \beta \in K \rightarrow \alpha x + \beta y \in \sum_{i=1}^k W_i$

$x \in \sum_{i=1}^k W_i \xrightarrow{def.} \exists x_1 \in W_1, \dots, x_k \in W_k$ t.d. $x = x_1 + \dots + x_k$

$y \in \sum_{i=1}^k W_i \xrightarrow{def.} \exists y_1 \in W_1, \dots, y_k \in W_k$ t.d. $y = y_1 + \dots + y_k$

$$\alpha x + \beta y = \alpha(x_1 + \dots + x_k) + \beta(y_1 + \dots + y_k) = \underbrace{(\alpha x_1 + \beta y_1)}_{\in W_1} + \dots + \underbrace{(\alpha x_k + \beta y_k)}_{\in W_k} \in \sum_{i=1}^k W_i$$

jer su to ptp. $\rightarrow \alpha x + \beta y \in \sum_{i=1}^k W_i$

$\rightarrow \boxed{\sum_{i=1}^k W_i \leq V}$

Kod sume $\forall x \in \sum_{i=1}^k W_i$ ima prikaz $x = \sum_{i=1}^k x_i$ (*), $x_i \in W_i$.

Ali nitko ne kaže da je prikaz jedinstven, tj. da su x_i, $i=1,\dots,k$ jedinst. odr. s x.

Def. **DIREKTNA SUMA** potprostora $W_1 \dotplus W_2 \dotplus \dots \dotplus W_k = \dot{\sum}_{i=1}^k W_i$ je svaka suma potprostora u kojoj je gornji prikaz (*) jedinstven, tj. $\forall x \in \dot{\sum}_{i=1}^k W_i$ $\exists!\, x_i \in W_i$ t.d. $x = x_1 + x_2 + \dots + x_k$.

Lema (TEST DA JE SUMA DIREKTNA)

Suma ptp. $W_1, \dots, W_k$ je direktna akko 0_V ima jedinstven prikaz! $(0_V = w_1 + \dots + w_k,\ w_i \in W_i \rightarrow \forall w_i = 0)$

DOKAZ:

$[\Rightarrow]$ suma direktna → svaki vektor iz sume ima $\exists!$ prikaz → 0_V ima $\exists!$ prikaz

$[\Leftarrow]$ Pokazujemo: 0_V jedinstven prikaz → svaki vektor jedin. prikaz ($\overset{def}{=}$ suma direktna)

Neka je $x = \sum_{i=1}^k W_i$ i neka ima dva (razl.) prikaza

$x = x_1 + \dots + x_k$
$x = x_1' + \dots + x_k'$ $\Big/ - \rightarrow 0_V = \underbrace{(x_1 - x_1')}_{\in W_1} + \dots + \underbrace{(x_k - x_k')}_{\in W_k} \rightarrow x_1 = x_1', \dots, x_k = x_k' \rightarrow$ suma direktna

(DZ) Neka je suma ptp. $W_1, \dots, W_k$ direktna. Tada:

1) $x_i \in W_i$, $x_i \neq 0$, $i = 1, \dots, k$, onda $\{x_1, \dots, x_k\}$ lin. nez. $\quad \alpha_1 x_1 + \dots + \alpha_k x_k = 0 \overset{\dot{+}}{\rightarrow} \alpha_1 \overset{\neq 0}{x_1} = \dots = \alpha_k \overset{\neq 0}{x_k} = 0$
$\rightarrow \alpha_i = 0 \quad \forall i = 1, \dots, k$
$\rightarrow \{x_1, \dots, x_k\}$ lin. nez.

2) baza za dir. sumu = baza W_1 + baza $W_2 + \dots +$ baza W_k (bolje $\cup$, a ne +)

baza $W_1 = \underbrace{\{x_1, \dots x_a\}}_{\text{lin. nez.}}$ $\quad$ baza $W_2 = \underbrace{\{x_b, \dots, x_c\}}_{\text{lin. nez.}}$ $\quad \dots \quad$ baza $W_k = \underbrace{\{x_e, \dots, x_k\}}_{\text{lin. nez.}}$

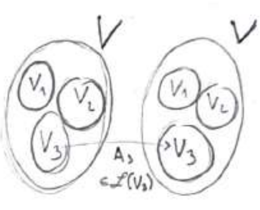
$\rightarrow$ svi x iz gornjeg redka su $\neq 0$ jer su elementi neke baze

$\rightarrow$ svi x iz gornjeg redka su međusobno lin. nez., ili jer su unutar iste baze ili zbog 1)

$\overset{\dot{+}}{\rightarrow}$ svaki v ima jedinstven prikaz u zadanoj sumi

$\Rightarrow$ (baza W_1, baza $W_2, \dots,$ baza W_k) = baza za W //

Još govorimo o A-invar. ptp.:



$A \in \mathcal{L}(V) \quad V_1, \dots, V_k$ A-invar. potprostori

Pretp. da je $V = V_1 \dot{+} \dots \dot{+} V_k$ (cijeli prostor je dir. suma A-invar. potprostora)

PRVO: $\quad V_i$ je A-invar. ptp. $\longrightarrow \overset{A_i :=}{A|V_i} \in \mathcal{L}(V_i)$

DRUGO: $\quad \forall x \in V \quad \exists! x_i \in V_i \quad i = 1, \dots, k \quad x = x_1 + \dots + x_k$

zbog invar.

Sada računamo: $\quad Ax = Ax_1 + \dots + Ax_k$

$Ax = A_1 x_1 + \dots + A_k x_k$

kratko pišemo: $\quad A = A_1 \dot{+} \dots \dot{+} A_k \quad$ A je dir. suma operatora A_i

ref. $t \geq 0 \Rightarrow A^t = A_1^t \dot{+} A_2^t \dot{+} \dots \dot{+} A_k^t \xrightarrow{\text{posljedica (kasnije u gradivu)}}$

Npr. $t = 0$ daje $I = A^0 = \underbrace{I_1}_{\text{na } V_1} \dot{+} \underbrace{I_2}_{\text{na } V_2} \dot{+} \dots \dot{+} \underbrace{I_k}_{\text{na } V_k}$ identitete

- jer za svaki pol. $f(A)$ lin. komb. tot $A^0, A_1, \dots \rightarrow f(A) = f(A_1) \dot{+} \dots \dot{+} f(A_k)$
- Usput: V_i A-invar. $\rightarrow$ $f(A)$-invar.
- prvo za $f(x) = x^t$ kada $f(A) = A^t$, a onda općenito

OBRATNO:

Ako imamo $V = V_1 \dot{+} \dots \dot{+} V_k$ te operatore $B_1 \in \mathcal{L}(V_1), \dots, B_k \in \mathcal{L}(V_k)$ onda $B := B_1 \dot{+} \dots \dot{+} B_k$ što po def. znači da je $Bx = B_1 x_1 + \dots + B_k x_k$ za $\forall x \in V$ napisan jedinstveno $x = x_1 + \dots + x_k$.

Tada vrijedi:
a) $B \in \mathcal{L}(V)$
b) B_i su B-invar.
c) B_i je $B|V_i$

"ovo možete sami provjeriti"

ovo raspisati

Vratimo se na početak:

$A \in \mathcal{L}(V) \quad V_1, \dots, V_k$ A-invar. ptp. t.d. $V = V_1 \dot{+} \dots \dot{+} V_k$

Baza za V = (baza $V_1, \dots,$ baza V_k)

MATR. ZAPIS:

$$A(\text{baza za } V) = \begin{bmatrix} \boxed{A_1} & & 0 \\ & \ddots & \\ 0 & & \boxed{A_k} \end{bmatrix} \quad \begin{matrix} \rightarrow \text{u odnosu na baza } V_1 \\ \\ \rightarrow \text{u odnosu na baza } V_k \end{matrix}$$

blok dij. matrica

Def. $N \in \mathcal{L}(V)$ je **NILPOTENTAN** ako $\exists k \geq 1$ t.d. $N^k = 0$.

Npr. $\boxed{\dim V = 1}$ $\quad V = K \cdot e, \quad e \in V \quad e \neq 0 \quad \rightarrow$ u ovom slučaju su samo 0_{op} nilpot.

V je isto što i K, $\mathcal{L}(V)$ množenje skalarom

$N(v) = \lambda v \quad \forall v \in V \quad \lambda \in K$

u ovom redu smo zapravo gledali "kada je N nilpot.?" $\leftarrow$ N je nilpot. jer $N^k = 0 \Rightarrow 0 = N^k(v) = \lambda^k(v) \quad \forall v \in V \rightarrow \lambda = 0 \rightarrow N = 0_{op}$

Npr. $\quad \dim V = 2 \quad e = (e_1, e_2)$ baza za V

$N(e) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \neq [0]$ $\quad$ nilpot. op. koji je zadan u bazi (e) matricom $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$N^2(e) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = [0] \rightarrow N^2 = 0_{op}$ na V

Vratimo se na opći slučaj:

Neka je $N \in \mathcal{L}(V)$ nilpot. $\rightarrow \exists k \geq 1$ t.d. $N^k = 0$

Imamo niz: $N^0 = I$

$N^1 = N$

$\vdots$ $\longleftarrow$ $\exists!$ najmanji $p \geq 1$ t.d. $N^{p-1} \neq 0_{op}$, $N^p = 0_{op}$, $N^{p+1} = 0_{op}$ ...

$N^k = 0_{op}$

$N^{k+1} = N^k \cdot N = 0 \cdot N = 0_{op}$

$N^{k+2} = 0_{op}$

$\vdots$

p - "indeks nilpotentnosti" nilpot. op. N

$\hookrightarrow$ ovisi o op. N

Npr. $N = 0_{op}$ je nilpot. op. indeksa nilpotentnosti $p = 1$, jer $0^1 = 0$, a $0^0 = I$.

CILJ: Opisati nilpotentne (SVE!) op. na V

STRATEGIJA: 1. slučaj: $p = \dim V$ (lakši)

2. slučaj: $p < \dim V$ (teži)

Tm. $N \in \mathcal{L}(V)$ nilpot. op. čiji indeks nilpot. je p.

Lema 3.1 1. $\forall v \in V$ t.d. $N^{p-1} \neq 0$ (takvih ima po def.) $\rightarrow \{v, Nv, \ldots, N^{p-1}v\}$ je lin. nez.

2. $\langle v, Nv, \ldots, N^{p-1}v \rangle$ (razapet vektorima $v, \ldots, N^{p-1}v$) je N-invar. ptp

3. $\dim \langle v, Nv, \ldots, N^{p-1}v \rangle = p$

Korolar 3.3 4. $p \leq \dim V$

DOKAZ.

① Pretp. da je $\alpha_0 v + \alpha_1 Nv + \ldots + \alpha_{p-1} N^{p-1} v = 0$ $\quad \alpha_0, \ldots, \alpha_{p-1} \in K$

$\hookrightarrow \sum_{i=0}^{p-1} \alpha_i N^i v = 0 \xrightarrow{/N^{p-1}} N^{p-1}\left(\sum_{i=0}^{p-1} \alpha_i N^i v\right) = N^{p-1} 0 = 0 \rightarrow \sum_{i=0}^{p-1} \alpha_i N^{p-1+i} v = 0$

GLAVNO U DOKAZU: $i \geq 1 \rightarrow p-1+i \geq p \rightarrow N^{p-1+i} = 0$ (Nul-op.)

To za posljedicu ima da na lijevoj strani "preživi" član za $i=0$, tj. $\alpha_0 N^{p-1} v = 0 \xrightarrow[\forall v \in V]{N^{p-1} \neq 0_{op}} \alpha_0 = 0$

Početna suma $\alpha_0 v + \alpha_1 Nv + \ldots + \alpha_{p-1} N^{p-1} v = 0$ glasi $\alpha_1 Nv + \ldots + \alpha_{p-1} N^{p-1} v = 0$, tj. $\sum_{i=1}^{p-1} \alpha_i N^i v = 0 \quad \xrightarrow{/N^{p-2}}$

"Napadnemo" sa $N^{p-2} \xrightarrow{\cdots} \alpha_1 = 0$ i tako ponavljamo $\xrightarrow{\cdots} \alpha_0 = \alpha_1 = \ldots = \alpha_{p-1} = 0 \rightarrow \boxed{\{v, Nv, \ldots, N^{p-1}v\} \text{ lin. nez.}}$

③ ① $\rightarrow$ $\boxed{\dim [\{v, Nv, \ldots, N^{p-1}v\}] = p}$

④ $[\{v, Nv, \ldots, N^{p-1}v\}] \leq V \rightarrow \boxed{p \leq \dim V}$

② Tvrdimo $\forall x \in [v, Nv, \ldots, N^{p-1}v] \rightarrow Nx \in [v, Nv, \ldots, N^{p-1}v]$

①

$\exists!\ \alpha_0, \alpha_1, \ldots, \alpha_{p-1} \in K$ t.d. $x = \alpha_0 v + \alpha_1 Nv + \ldots + \alpha_{p-1} N^{p-1} v \quad / \overset{\text{l.o.}}{N}$

$Nx = \alpha_0 Nv + \alpha_1 N^2 v + \ldots + \alpha_{p-2} N^{p-1} v + \alpha_{p-1} \underbrace{\overset{0_{op}}{N^p v}}_{0}$

$Nx = \alpha_0 Nv + \alpha_1 N^2 v + \ldots + \alpha_{p-2} N^{p-1} v \in [\{v, Nv, \ldots, N^{p-1}v\}] \rightarrow \boxed{[\{v, Nv, \ldots, N^{p-1}v\}] \text{ jest } N\text{-invar.}}$

DZ Kor. 3.4.

Kor. Neka je $N \in \mathcal{L}(V)$ nilpot. op. ind $= p$. Neka je $v \in V$ proizvoljan vektor koji zadovoljava $N^{p-1} v \neq 0$.

Tada vrijedi: $N([\{v, Nv, \ldots, N^{p-1}v\}]) = [\{Nv, N^2 v, \ldots, N^{p-1}v\}]$ je $\dim = p-1$

$N^2([\{v, Nv, \ldots, N^{p-1}v\}]) = [\{N^2 v, N^3 v, \ldots, N^{p-1}v\}]$ je $\dim = p-2$

$\vdots$

$N^{p-1}([\{v, Nv, \ldots, N^{p-1}v\}]) = [\{N^{p-1}v\}]$ je $\dim = 1$

$N^k([\{v, Nv, \ldots, N^{p-1}v\}]) = [\{0_V\}]$ je $\dim = 0$ za $k \geq p$

DOKAZ: $\{v, Nv, \ldots, N^{p-1}v\}$ lin. nez.

$\hookrightarrow \{Nv, \ldots, N^{p-1}v\}$ lin. nez

$\rightarrow \boxed{\dim [\{Nv, \ldots, N^{p-1}v\}] = p-1}$

(ostale tvrdnje slično)

Razmotrimo slučaj $p = \dim V$ (koji je korolar prethodnog tm.)

Kor. Neka je $N \in \mathcal{L}(V)$ nilpot. op. t.d. $p = \dim V$ ("punog indeksa"). Tada $\forall v \in V$ t.d. $N^{p-1} v \neq 0$ vrijedi $e := (\underbrace{N^{p-1}v}_{e_1}, \underbrace{N^{p-2}v}_{e_2}, \ldots, \underbrace{Nv}_{e_{p-1}}, \underbrace{v}_{e_p})$ je baza za V te vrijedi: $Ne_1 = N(N^{p-1}v) = N^p v = 0$, $Ne_2 = N(N^{p-2}v) = N^{p-1}v = e_1$, ... $Ne_p = Nv = e_{p-1}$

Matrica $N(e) = \begin{bmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & 1 \\ 0 & & & 0 \end{bmatrix}$ → "elementarna Jordanova klijetka dim. p." J_p

(Ne_1 ... Ne_p)

DOKAZ: Slijedi odmah iz Tm-a., uz napomenu $p = \dim V \rightarrow [\{v, Nv, \ldots, N^{p-1}v\}] = V \rightarrow (N^{p-1}v, \ldots, Nv, v)$ baza za V

! (Nap) Baza $(v, Nv, \ldots, N^{p-1}v)$ "nije dobra"

Primjeri Jord. klijetki: $J_1 = [0]$ - nema "ništa" iznad dijagonale - to je situacija 1. primjera ($\dim V = 1$)

$J_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\quad J_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad J_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

ponavljanje

$N \in \mathcal{L}(V)$ nilpotentan indeksa p, što znači $N^{p-1} \neq 0_{op}$ $N^p = 0_{op}$

Osnovni slučaj je $p = \dim V$

Izaberemo $0 \neq v \in V$ t.d. $N^{p-1}v \neq 0_v \to (N^{p-1}v, \dots, Nv, v)$ baza za V u kojoj je matrica operatora N dana sa: $J_p = \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{bmatrix}$

Lema **TEHNIČKA LEMA** (dio dokaza 3.7.)

Neka je $v \in V$ t.d. $N^{p-1}v \neq 0$. Konstruiramo ciklički ptp $V_1 = \langle N^{p-1}v, \dots, Nv, v \rangle$ (moguće $V_1 \leq V$).
Tada postoji N-invarijantan ptp. $W_1 \leq V$, t.d. $V = V_1 \dotplus W_1$

OBJAŠNJENJE: $\forall w \in W_1 \to Nw \in W_1$ te možemo definirati operator $N|W_1 \in \mathcal{L}(W_1)$
Očito $(N|W_1)^p = N^p|W_1 = 0_{op}|W_1 = 0_{op} \to N|W_1$ je nilpot i ind. nilpot $=: q$
$(N|W_1)^{q-1} \neq 0_{op}$, $(N|W_1)^q = 0_{op}$, vrijedi $q \leq p$
(tj. dobili smo opet nilpot. operator indeksa $\leq p$.)
Također V N-inv. (dokazali na prošlom pred.) i vrijedi $N|V_1$ nilpot. indeksa p

IDEJA DOKAZA: (neće pitati dokaz)

(1.) $N \in \mathcal{L}(V)$ te konstruiramo adjung. op. $N' \in \mathcal{L}(V', V')$ zadan s $(N'g)(w) = g(Nw)$ $\forall g \in V'$, $\forall w \in V$
Pokaže se da je N' nilpot. ind. $= p$. Dakle, te zato vrijedi (2.)

(2.) Postoji funkcional $f \in V^*$ t.d. $(N')^{p-1}(f)(v) \neq 0$

(3.) Konstruiramo ciklički ptp. u V^* (V') $\overline{W_1} := \langle (N')^{p-1}(f), \dots, N'f, f \rangle$ te možemo uzeti W_1: anihilator od $\overline{W_1}$ u V

Iz tehničke leme indukcijom nalazimo sljedeće:

Tm. Neka je $N \in \mathcal{L}(V)$ nilpotentan operator indeksa p. Tada postoji dekompozicija u direktnu sumu $V = V_1 \dotplus V_2 \dotplus \dots \dotplus V_m$, gdje su:
1.) $V_1, V_2, \dots, V_m$ N-invar. ptp.
2.) $p_i := \dim V_i$, $i = 1, \dots, m \to N_i := N|V_i$ je nilpotentan indeksa p
3.) $p_1 \geq p_2 \geq \dots \geq p_m$
4.) $p = p_1$ (što odmah vidimo iz tehničke leme)

DOKAZ: Slijedi iz tehničke leme.

Korolar 1. $N = N_1 \dotplus N_2 \dotplus \dots \dotplus N_m$
(što znači: $\forall x \in V$ napišemo $x = x_1 + \dots + x_m$ $x_i \in V_i \to Nx = N_1x_1 + \dots + N_mx_m$)

Nap. Uočimo da (2.) pokazuje da svaki od N_i pripada osnovnom slučaju (nilpot. ind. $= p = \dim V$) te izborom vektora $v_i \in V_i$ t.d. je $N_i^{p_i-1}v_i \neq 0$ imamo cikličku bazu $(N_i^{p_i-1}v_i, \dots, N_iv_i, v_i)$ za V_i u kojoj matrica operatora N_i je J_{p_i}.

Korolar 2. Objedinjavanjem baze iz gornje napomene, za $i = 1, \dots, m$ dobijemo bazu za V u kojoj je matrica operatora N (ponovimo opći slučaj) blok dijag. matrica: $\begin{bmatrix} J_{p_1} & & 0 \\ & \ddots & \\ 0 & & J_{p_m} \end{bmatrix}$

Pitanje: Koliko je dekompozicija iz tm. jedinstvena?

Odgovor: Nije jedinstvena, ali "čudo se desi" matrični zapis jest jedinstven! Zašto?
$\forall p_1, p_2, \dots, p_m \in \{1, 2, \dots, p\}$ Možemo se pitati: za dani $k \in \{1, 2, \dots, p\}$ koliko ima indeksa $i \in \{1, \dots, m\}$ t.d. $p_i = k$.
ODGOVOR: Tm. $\boxed{\#\{i; p_i = k\} = r(N^{k+1}) + r(N^{k-1}) - 2r(N^k) =: n_k}$ → ovo mi pokazuje koliko puta (n_k) se javlja prostor (klijetka) veličine k

Nap. $r(A) = \dim\{Av; v \in V\}$ rang op. A

O matričnom zapisu možemo razmišljati ovako:
$\nu_k := r(N^{k+1}) + r(N^{k-1}) - 2r(N^k)$ $k = 1, \dots, p$
Sada matr. zapis od N u bazi koju smo ranije spomenuli:

$$\begin{bmatrix} J_p & & & & & & \\ & \ddots & & & & & \\ & & J_p & & & & \\ & & & J_{p-1} & & & \\ & & & & \ddots & & \\ & & & & & J_{p-1} & \\ & & & & & & \ddots \\ & & & & & & & J_1 \\ & & & & & & & & \ddots \\ & & & & & & & & & J_1 \end{bmatrix}$$

ν_p sigurno ≥ 1
ν_{p-1}
ako $\nu_{p-1} = 0$ izpuštamo J_{p-1}

Što je isto kao: $\begin{bmatrix} J_{p_1} & & \\ & \ddots & \\ & & J_{p_m} \end{bmatrix}$.

DZ zadnji tm. u ovoj cjelini: (1.) $k_N(x) = x^p$ (2.) $\sigma_N = \{0\}$
DOKAZ – treba znati

4. POLUPROST I OPERATORI

(dijagonalizabilni)

- "drastična" razlika u slučajevima $K=\mathbb{R}$ i $K=\mathbb{C}$
- mi razmatramo $K=\mathbb{C}$ i to pretpostavljamo

Teorija nilpotentnog operatora ne ovisi o izboru realnog ili kompl. v.p.

Od sada (do 1. kol.) SVE RADIMO U $\mathbb{C}$!

Def. Za $A \in \mathcal{L}(V)$ kažemo da je POLUPROST ako postoji baza $e=(e_1,\dots,e_n)$ za V i $\mu_1,\dots,\mu_n \in \mathbb{C}$ t.d. $A_{(e)} = \begin{bmatrix} \mu_1 & & 0 \\ & \ddots & \\ 0 & & \mu_n \end{bmatrix}$, tj. A se može dijagonalizirati u bazi (e).

Prop. U oznakama iz definicije, vrijedi:

① $k_A(x) = (x-\mu_1)(x-\mu_2)\cdot\ldots\cdot(x-\mu_n)$

② $\sigma_A = \{\mu_1,\dots,\mu_n\}$

DOKAZ:

① $k_A(x) = \det(xI-A) \overset{(e)}{=} \det(xI-A_{(e)}) = \begin{vmatrix} x-\mu_1 & & 0 \\ & \ddots & \\ 0 & & x-\mu_n \end{vmatrix} = (x-\mu_1)\ldots(x-\mu_n)$ ✓

② $\sigma_A = \{\lambda \in K;\ k_A(\lambda)=0\} = \{\lambda \in K;\ (\lambda-\mu_1)\ldots(\lambda-\mu_n)\} = \{\mu_1,\mu_2,\dots,\mu_n\}$ ✓

Iduće nas zanima $\mu_A = ?$

Za odgovoriti na to pitanje, moramo malo bolje "organizirati" $\mu_1,\dots,\mu_n$!

1. Neka su $\lambda_1,\lambda_2,\dots,\lambda_s$ svi mogući međusobno različiti između $\mu_1,\dots,\mu_n$.
2. Za svaki od $\lambda_1,\dots,\lambda_s$ pogledamo sve i t.d. $\mu_i=\lambda_j$ i presložimo bazu da dobijemo novu bazu, tj. $\forall k \in \{1,\dots,s\}$ pogledamo λ_k i skup $\{i;\ \mu_i=\lambda_k\}$ te definiramo $V_k :=$ razapeto svim e_i t.d. $\mu_i = \lambda_k$

Npr.:

$n=5$ $\mu_1=1$ $\mu_2=2$ $\mu_3=1$ $\mu_4=1$ $\mu_5=3$

$s=3$ $\lambda_1=1$ $\lambda_2=2$ $\lambda_3=3$

$\lambda_1=1 \to \{i;\ \mu_i=\lambda_1\}=\{1,3,4\}$ $V_1=[\{e_1,e_3,e_4\}]$

$\lambda_2=2 \to \{i;\ \mu_i=\lambda_2\}=\{2\}$ $V_2=[\{e_2\}]$

$\lambda_3=3 \to \{i;\ \mu_i=\lambda_3\}=\{5\}$ $V_3=[\{e_5\}]$

Kako je $\{e_1,e_2,e_3,e_4,e_5\}$ baza za V $\xrightarrow{\text{ODMAH}}$ $V=\underset{\langle e_1,e_3,e_4\rangle}{V_1} \dotplus \underset{\langle e_2\rangle}{V_2} \dotplus \underset{\langle e_5\rangle}{V_3}$ Dobili bazu (e_1,e_2,e_3,e_4,e_5) za V.

Ista stvar vrijedi općenito: $V=V_1\dotplus\dots\dotplus V_s$

Pogledajmo malo detaljnije V_k.

V_k je razapet svim e_i t.d. $\mu_i=\lambda_k$. A što to znači?

Takav i zadovoljava $Ae_i \overset{\text{zbog sliježi iz dijag. forme}}{=} \mu_i e_i = \lambda_k e_i \to (A-\lambda_k I)e_i = 0$

$\to e_i$ se nalazi unutar sv.ptp. $\mathrm{Ker}(A-\lambda_k I)$

$\to V_k$ kao njihova ljuska je ptp. $\mathrm{Ker}(A-\lambda_k I) = \mathrm{Ker}(\lambda_k I - A)$

$(A-\lambda_k I)e_i=0 \to e_i \in \mathrm{Ker}(A-\lambda_k I) \to V_k \le \mathrm{Ker}(A-\lambda_k I)$

dovoljno je pitati ...

Iz direktne sume i činjenice da su $\lambda_1,\dots,\lambda_s$ međusobno različiti $\to V_k = \mathrm{Ker}(A-\lambda_k I)$ $k=1,\dots,s$

Sve skupa:

Korolar Uz oznake iz definicije vrijedi:

① $\sigma_A = \{\mu_1,\dots,\mu_n\} = \{\lambda_1,\dots,\lambda_s\}$ - jer uzimamo samo međusobno različite

② $V = \dot{+}_{k=1}^{s} N(A-\lambda_k I)$ "N" := "Ker"

③ $A = A_1 \dotplus \dots \dotplus A_k = \lambda_1 I_1 \dotplus \dots \dotplus \lambda_s I_s$

OBJAŠNJENJE: A-invar. i vrijedi $\underbrace{A|_{N(A-\lambda_k I)}}_{A_k} = \lambda_k \cdot I_k$ ↳ jedinični na $N(A-\lambda_k I)$

VAŽNO! Odgovor na pitanje što je μ_A?

Iz § 2 $\to \sigma_A = \{x \in \mathbb{C};\ \mu_A(x)=0\}$ (od ranije, iz korolara ①) $=\{\lambda_1,\dots,\lambda_s\}$

$\forall k \in \{1,\dots,s\} \to \mu_A(\lambda_k)=0 \to x-\lambda_k \mid \mu_A(x)$ (dijeljenje pol.)

Kako su $\lambda_1,\dots,\lambda_s$ međusobno različiti $\to (x-\lambda_1)(x-\lambda_2)\cdot\ldots\cdot(x-\lambda_s) \mid \mu_A(x)$

Nazovimo $f(x) = (x-\lambda_1)(x-\lambda_2)\cdot\ldots\cdot(x-\lambda_s)$

Vidimo: f normiran (vodeći koef. $=1$), $f \mid \mu_A$

Tm. $K=\mathbb{C}$, $A \in \mathcal{L}(V)$

A poluprost

$\Updownarrow$

μ_A ima jednostruke korijene

"KARAKTERIZACIJA POLUPROSTOG OPERATORA"

Korolar $\mu_A = f$

DOKAZ: Prema korak. min pol. i gornjeg, dovoljno je $f(A)=0$ a to slijedi iz Kor ③.

$A = \dot{+}_{k=1}^{s} \lambda_k I_k \xrightarrow{\S 2} f(A) = \dot{+}_{k=1}^{s} f(\lambda_k I_k) = (\lambda_k I_k - \lambda_1 I_k)(\lambda_k I_k - \lambda_2 I_k)\cdot\ldots\cdot(\lambda_k I_k - \lambda_s I_k) \to$ pojavljuje se $(\lambda_k I_k - \lambda_k I_k)=0$

Indeksi mogu zbunjivati. Ako $F(x)=(x-\lambda_1)\ldots(x-\lambda_s)$ $\forall A \in \mathcal{L}(V)$ $F(A)=(A-\lambda_1 I)\cdot\ldots\cdot(A-\lambda_s I)$

$F(B)=(B-\lambda_1 I_k)\ldots(B-\lambda_s I_k)$ te ako $B:=\lambda_k I_k$ dobijemo što treba

Korolar μ_A ima jednostruke korijene

5. JORDANOVA FORMA

Nap. Gledamo samo kompleksne v.p., tj. $\mathbb{K} = \mathbb{C}$!

$A \in \mathcal{L}(V) \rightarrow k_A(x) = \det(xI - A) = (x-\lambda_1)^{n_1}(x-\lambda_2)^{n_2} \cdot \ldots \cdot (x-\lambda_s)^{n_s}$

$\lambda_1, \lambda_2, \ldots, \lambda_s$ međusobno različite sv. vr. (zajedno čine sve moguće sv. vr. op A)

$\sigma_A = \{\lambda \in \mathbb{C};\ \lambda I - A$ nije invert.$\} = \{\lambda_1, \lambda_2, \ldots, \lambda_s\}$

$\lambda I - A$ nije invert. $\Leftrightarrow \det(\lambda I - A) = 0 \Leftrightarrow k_A(\lambda) = 0 \Leftrightarrow \lambda \in \{\lambda_1, \ldots, \lambda_s\}$

$\xrightarrow{§2} \mu_A(x) = (x-\lambda_1)^{p_1}(x-\lambda_2)^{p_2} \cdot \ldots \cdot (x-\lambda_s)^{p_s} \qquad 1 \le p_1 \le n_1, \ldots, 1 \le p_s \le n_s$

Ponovimo: A nilpotentan $\overset{§3}{\Leftrightarrow}$ $s=1$ $\lambda_1 = 0$ tj. $\mu_A(x) = x^q$

A poluprost $\overset{§4}{\Leftrightarrow}$ $p_1 = p_2 = \cdots = p_s = 1$, tj. min. pol. ima samo jednostruke nultočke

Trebaju nam A-invarijantni ptp. (W je A-inv. ako $\forall w \in W \rightarrow Aw \in W$, te možemo def. $A|_W \in \mathcal{L}(W)$)

Jednostavna procedura konstrukcije (koja ne daje sve):

$\lambda \overset{\text{sv.vr.}}{\in} \mathbb{C} \quad m = 1,2,3,\ldots \quad (\lambda I - A)^m \in \mathcal{L}(V)$ za koji $\mathrm{Ker}((\lambda I - A)^m) := \{v \in V;\ (\lambda I - A)^m v = 0\}$

Lema 1. Označimo $V_{\lambda,m,A} := V_{\lambda,m} := \mathrm{Ker}((\lambda I - A)^m)$

a) $V_{\lambda,m}$ je A-inv. $\forall \lambda \in \mathbb{C}\ \forall m = 1,2,\ldots$

b) $V_{\lambda,m} \subseteq V_{\lambda,m+1}\ \forall m \ge 1$ tj. $V_{\lambda,1} \subseteq V_{\lambda,2} \subseteq \ldots$

c) $V_{\lambda,m} = \{0\}$ ako $\lambda \ne \lambda_1, \lambda_2, \ldots, \lambda_s$

d) $V_{\lambda_i,1}$ je sv. ptp. za sve sv. vr. λ_i

$A|_{V_{\lambda,m}}$

DOKAZ:

a) Želimo pokazati $(\lambda I - A)^m A = A(\lambda I - A)^m$ u $\mathcal{L}(V)$.

$f(x) := (\lambda - x)^m \cdot x = x \cdot (\lambda - x)^m \xrightarrow{\text{evaluacijom}} f(A) = (\lambda I - A)^m A = A(\lambda I - A)^m \rightarrow$ komutativnost ✓

$v \in V_{\lambda,m} \rightarrow (\lambda I - A)^m v = 0$ (po def) $\rightarrow A(\lambda I - A)^m v = A0 = 0$

$(\lambda I - A)^m A(v) = 0 \rightarrow A(v) \in V_{\lambda,m} \rightarrow V_{\lambda,m}$ je A-invar. //

b) $v \in V_{\lambda,m} \rightarrow (\lambda I - A)^m v = 0 \rightarrow (\lambda I - A)[(\lambda I - A)^m v] = (\lambda I - A)0 = 0$

$\rightarrow (\lambda I - A)^{m+1} v = 0 \rightarrow v \in V_{\lambda,m+1} \rightarrow \boxed{V_{\lambda,m} \subseteq V_{\lambda,m+1}}$ //

c) $\lambda \ne \lambda_1, \ldots, \lambda_s \Leftrightarrow k_A(\lambda) \ne 0 \Leftrightarrow \det(\lambda I - A) \ne 0$

$\rightarrow \det((\lambda I - A)^m) = (\det(\lambda I - A))^m \ne 0 \rightarrow (\lambda I - A)^m$ je invertibilan $\rightarrow \boxed{V_{\lambda,m} = \mathrm{Ker}((\lambda I - A)^m) = \{0\}}$

zašto?

d) Ako je $\lambda = \lambda_i$ za neki $i = 1, \ldots, s$

Onda $V_{\lambda_i,1} = \mathrm{Ker}((\lambda_i I - A)) \ni v \Leftrightarrow (\lambda_i I - A)v = 0 \Leftrightarrow \lambda_i I v - Av = 0 \Leftrightarrow \lambda_i v - Av = 0$

$\ne 0$ je op. $\lambda I - A$ nije invert. jer $\det(\lambda_i - A) = k_A(\lambda_i) = 0$

$\Leftrightarrow Av = \lambda_i v$

$\rightarrow V_{\lambda_i,m} \ne \{0\}$ zbog (b) jer $V_{\lambda_i,1} \subseteq V_{\lambda_i,m}$ $m \ge 1$

Lema 2. 1) Neka su $m_1, m_2, \ldots, m_s \ge 1$ dani. Tada je suma ptp. V_{λ_i,m_i} $1 \le i \le s$ direktna, što pišemo $V_{\lambda_1,m_1} \dotplus V_{\lambda_2,m_2} \dotplus \ldots \dotplus V_{\lambda_s,m_s}$.

2) Ako je $m_1 \ge p_1, \ldots, m_s \ge p_s$ (kratnosti u min.pol.), onda vrijedi $V_{\lambda_1,m_1} \dotplus \ldots \dotplus V_{\lambda_s,m_s} = V$

3) $V_{\lambda_1,p_1} \dotplus \ldots \dotplus V_{\lambda_s,p_s} = V$ naziv: "korijeni potprostori za A" !

DOKAZ: naprav je dugačak i težak dokaz kojega neće pitati pa nisam prepisala iz bilj.

5. JORDANOVA FORMA ($\mathbb{C}$)

$A \in \mathcal{L}(V) \rightarrow k_A(x) := \det(xI - A) = (x-\lambda_1)^{n_1} \cdots (x-\lambda_s)^{n_s}$ $\quad \lambda_1, \dots, \lambda_s$ međusobno rasl. sv. vr.

$\sigma_A := \{\lambda \in \mathbb{C};\ \lambda I - A$ nije invert.$\} = \{\lambda \in \mathbb{C}; \det(\lambda I - A) = 0\} = \{\lambda \in \mathbb{C}; k_A(\lambda) = 0\} = \{\lambda_1, \dots, \lambda_s\}$

$\rightarrow \mu_A(x) = (x-\lambda_1)^{p_1} \cdots (x-\lambda_s)^{p_s} \quad 1 \le p_i \le n_i$

A nilpotentan $\leftrightarrow s = 1 \wedge \lambda_1 = 0$ tj. $\mu_A(x) = x^{p_1}$ ind. nilpot. od A

A poluprost $\leftrightarrow p_1 = \dots = p_s = 1$ t. min. pol. ima samo jednostruke nultočke

Trebamo A-invar. ptp. Jednostavna konstrukcija: $\lambda \in \mathbb{C}$ $m = 1, 2, \dots$ $(\lambda I - A)^m \in \mathcal{L}(V)$ $\mathrm{Ker}((\lambda I - A)^m) = \{v \in V;\ (\lambda I - A)^m v = 0\}$

Lema: Označimo $V_{\lambda,m,A} := V_{\lambda,m} := \mathrm{Ker}((\lambda I - A)^m)$.
a) $V_{\lambda,m}$ je A-invar. $\forall \lambda \in \mathbb{C}\ \forall m = 1, 2, \dots$
b) $V_{\lambda,m} \subseteq V_{\lambda,m+1}\ \forall m \ge 1$
c) $V_{\lambda,m} = \{0\}$ ako $\lambda \ne \lambda_1, \dots, \lambda_s$
d) $V_{\lambda_i,1}$ je sv. ptp. za sve λ_i

$\mathrm{Ker}((\lambda I - A)^m)$
a) $\mathrm{Ker}((\lambda I - A)^m)$ je A inv $\forall \lambda \in \mathbb{C}\ \forall m = 1, 2, \dots$
b) $\mathrm{Ker}((\lambda I - A)^m) \subseteq \mathrm{Ker}((\lambda I - A)^{m+1})$ $\forall m \ge 1$
c) $\mathrm{Ker}((\lambda I - A)^m) = \{0_V\}$ ako $\lambda \notin \sigma_A$
d) $\mathrm{Ker}((\lambda_s I - A)^1)$ je sv. ptp. za $\lambda_s \in \sigma$

DOKAZ
a) $f(x) := (\lambda - x)^m x = x \cdot (\lambda - x)^m \xrightarrow{\text{eval.}} (\lambda - A)^m A = A(\lambda - A)^m$ kom. ✓
$v \in V_{\lambda,m} \rightarrow (\lambda I - A)^m v = 0 \Rightarrow A(\lambda I - A)^m v = (\lambda I - A)^m A v = 0 \rightarrow Av \in \mathrm{Ker}(\lambda I - A)^m \rightarrow \boxed{V_{\lambda,m}\ \text{A-invar.}}$

b) $v \in V_{\lambda,m} \rightarrow (\lambda I - A)^m v = 0 \rightarrow (\lambda I - A)[(\lambda I - A)^m v] = 0 \rightarrow v \in V_{\lambda,m+1} \rightarrow \boxed{V_{\lambda,m} \subseteq V_{\lambda,m+1}}$

c) $\lambda \ne \lambda_1, \dots, \lambda_s \leftrightarrow k_A(\lambda) \ne 0 \leftrightarrow \det(\lambda I - A) \ne 0$
$\det((\lambda I - A)^m) = (\det(\lambda I - A))^m \ne 0 \leftrightarrow (\lambda I - A)^m$ reg. (invert.) $\rightarrow \boxed{\mathrm{Ker}(\lambda I - A)^m = \{0_V\}}$

d) $\lambda = \lambda_i \quad i = 1, \dots, s \rightarrow V_{\lambda_i,1} = \mathrm{Ker}(\lambda_i I - A)$
$v \in \mathrm{Ker}(\lambda_i I - A) \leftrightarrow (\lambda_i I - A)v = 0 \leftrightarrow \lambda_i I v = Av \leftrightarrow Av = \lambda_i v \rightarrow \boxed{V_{\lambda_i,1}\ \text{sv. ptp.}}$
$V_{\lambda_i,m} \ne \{0\}$ zbog b) jer $V_{\lambda_i,1} \subseteq V_{\lambda_i,m}$ $m \ge 1$

Lema 1) Neka su $m_1, \dots, m_s \ge 1$ dani, tada je suma ptp. V_{λ_i,m_i}, $1 \le i \le s$ direktna, $V_{\lambda_1,m_1} \dot{+} \dots \dot{+} V_{\lambda_s,m_s}$
2) Ako je $m_1 \ge p_1, \dots, m_s \ge p_s$ (kratnosti u min. pol.), onda $V_{\lambda_1,m_1} \dot{+} \dots \dot{+} V_{\lambda_s,m_s} = V$
3) $V_{\lambda_1,p_1} \dot{+} \dots \dot{+} V_{\lambda_s,p_s} = V$ naziv: "KORIJENI PTPROSTORI ZA A"

DOKAZ - neće pitati

$\sigma_A = \{\lambda_1, \dots, \lambda_s\} \ni \lambda \leftrightarrow \overbrace{\mathrm{Ker}(\lambda I - A)}^{V_{\lambda,1}\ \text{sv. ptp.}} \ne \{0\}$

Def. KORIJENI PTP $\mathrm{Ker}(\lambda I - A)^{p_s}$ (sv. vr.; kr. u. μ_A)

Lema 1 $\lambda \in \sigma \rightarrow V_{\lambda,1} \subseteq V_{\lambda,2} \subseteq \dots$
$V_{\lambda,m} := \mathrm{Ker}((\lambda I - A)^m) \rightarrow A|V_{\lambda,m} \in \mathcal{L}(V_{\lambda,m})$

Lema 3 $\overbrace{V_{\lambda,1}}^{\rightarrow\text{sv. ptp.}} \subseteq V_{\lambda,2} \subseteq \dots \subseteq \overbrace{V_{\lambda,p}}^{\rightarrow\text{korijeni ptp.}} = V_{\lambda,p+1} = \dots$ $\quad p = p_i$ ak. $\lambda = \lambda_i$

Lema 2 KLJUČNA TEHNIČKA LEMA
Ako su $m_1 \ge p_1, \dots, m_s \ge p_s \rightarrow V = V_{\lambda_1,m_1} \dot{+} \dots \dot{+} V_{\lambda_s,m_s}$ (dio leme)

Nap. $V_{\lambda,m} = \mathrm{Ker}((\lambda I - A)^m) = \mathrm{Ker}((A - \lambda I)^m)$

TM $K = \mathbb{C}$, $A \in \mathcal{L}(V)$, $V_i = V_{\lambda_i,p_i}$ $i = 1, \dots, s$ (korijeni ptp. za λ_i) ($V_i = \mathrm{Ker}(A - \lambda_i I)^{p_i}$ $i = 1, \dots, s$).
Tada $V = V_1 \dot{+} \dots \dot{+} V_s$.
Nadalje: a) $\mathrm{Ker}(\lambda_i I - A) \subseteq V_i$ sv. ptp. $\subseteq$ kor. ptp.
b) $V_i = V_{\lambda_i,k} = \mathrm{Ker}(A - \lambda_i I)^k$ $k \ge p_i$ $\forall i$
c) $\mathrm{Ker}(A - \lambda_i I)^k = V_{\lambda_i,k} \subsetneq V_i$ $1 \le k < p_i$ $\forall i$
d) $\dim V_i = n_i$ (n_i = kratnost nult. u sv. pol.)
e) $A_i := A|V_i \in \mathcal{L}(V_i)$ Tada $A = A_1 \dot{+} \dots \dot{+} A_s$
$\mu_{A_i}(x) = (x - \lambda_i)^{p_i}$ $\forall i$ te zato $\mu_A(x) = \mu_{A_1}(x) \cdots \mu_{A_s}(x)$

a) $\mathrm{Ker}(\lambda_i I - A) \subseteq \mathrm{Ker}(\lambda_i I - A)^{p_i}$
b) $\mathrm{Ker}(\lambda_i I - A)^{p_i} = \mathrm{Ker}(\lambda_i I - A)^k$ $k \ge p_i$
c) $\mathrm{Ker}(\lambda_i I - A)^k \subsetneq \mathrm{Ker}(\lambda_i I - A)^{p_i}$ $k < p_i$
d) $\dim(\mathrm{Ker}(\lambda_i I - A)^{p_i}) = n_i$
e) $A_i = A|\mathrm{Ker}(\lambda_i I - A)^{p_i}$
$A = A_1 \dot{+} \dots \dot{+} A_s$
$\mu_{A_i}(x) = (x - \lambda_i)^{p_i}$
$\mu_A(x) = \mu_{A_1}(x) \cdots \mu_{A_s}(x)$

★ Pokazujemo $\overset{*}{=} 0$
$A = A_1 \dot{+} \dots \dot{+} A_s \rightarrow f(A) = f(A_1) \dot{+} \dots \dot{+} f(A_s)$
Pokazujem $f(A_i) = 0_{op}$ $\forall i = 1, \dots, s$
$f(x) = (x - \lambda_1)^{p_1} \cdots (x - \lambda_s)^{p_s}$
$= \underbrace{\Big(\prod_{\substack{j=1 \\ j \ne i}}^{s} (x - \lambda_j)^{m_j}\Big)}_{:= g_i(x)} (x - \lambda_i)^{m_i}$
$\rightarrow f(x) = g_i(x) \cdot (x - \lambda_i)^{m_i}$
$f(A_i) = g_i(A_i) \underbrace{(A_i - \lambda_i I)^{m_i}}_{=0} = 0 \rightarrow \boxed{f(A) = 0}$
jer $\mu_{A_i}(x) = (x - \lambda_i)^{m_i}$ i $\mu_{A_i}(A_i) = 0$

DOKAZ
• $V_i = \mathrm{Ker}(A - \lambda_i I)^{p_i}$ iz Leme 2 za $m_i = p_i \rightarrow V = V_1 \dot{+} \dots \dot{+} V_s$

a) $V_{\lambda_i,1} \subseteq V_{\lambda_i,p_i}$ Lema 1

b) $V_{\lambda_i,p_i} = V_{\lambda_i,k}$ Lema 3

c) $i \in \{1, \dots, p_i\}$ $1 \le k < p_i$
Ako bi bilo $V_i = V_{\lambda_i,k} = \mathrm{Ker}((\lambda_i I - A)^k)$
$\forall y \in V_i \rightarrow (\lambda_i I - A)^k y = 0 \ /\cdot(-1)^k \rightarrow (A - \lambda_i I)^k y = 0 \xrightarrow{\text{rest.}} (A_i - \lambda_i I_i)^k y = 0$
$(A_i - \lambda_i I_i)^k = 0_{op} \rightarrow A_i$ poništ. pol. $(x - \lambda_i)^k \rightarrow \mu_{A_i}(x) \overset{*}{=} (x - \lambda_i)^{p_i} | (x - \lambda_i)^k$
$\rightarrow p_i \le k$ ↯ $\rightarrow \boxed{V_{\lambda_i,k} \subsetneq V_i}$

d) e) $\rightarrow \mu_{A_i}(x) = (x - \lambda_i)^{p_i}$, $A_i \in \mathcal{L}(V_i)$
$\sigma_A = \{\lambda_i\}$, $k_A(x) = (x - \lambda_i)^{k_i}$ $p_i \le k_i$

e) $A = A_1 \dot{+} \dots \dot{+} A_s$ jasno jer to slijedi po def.
$V = V_1 \dot{+} \dots \dot{+} V_s$ znači $\forall x \in V$ $\exists!$ $x_1 \in V_1, \dots, x_s \in V_s$ t.d. $x = x_1 + \dots + x_s$
$\rightarrow Ax = Ax_1 + \dots + Ax_s$ —(situacija u V_i)→ $Ax = A_1 x_1 + \dots + A_s x_s \rightarrow A = A_1 \dot{+} \dots \dot{+} A_s$
CILJ: $\forall i = 1, \dots, s$ $(x - \lambda_i)^{p_i}$ je min. pol. od A_i $\quad$ djeluje na $x \in V_i$ te se zato ponaša kao $A_i = A|V_i$, isto kao identiteta I na V kada $I|V_i = I_i$ "ident. na V_i" te zato piše $(A_i - \lambda_i I_i)^{p_i} y = 0$ $\forall y \in V_i$ $\rightarrow (A_i - \lambda_i I_i)^{p_i} = 0$
$V_i = V_{\lambda_i,p_i} = \mathrm{Ker}((\lambda_i I - A)^{p_i}) \rightarrow \forall y \in V_i$ $(\lambda_i I - A)^{p_i} y = 0 \xrightarrow{(-1)^{p}} (A - \lambda_i I)^{p_i} y = 0$
Zadnje pokazuje: A_i poništava $(x - \lambda_i)^{p_i}$ $\rightarrow \mu_{A_i} | (x - \lambda_i)^{p_i}$ te nema druge nego $\mu_{A_i}(x) = (x - \lambda_i)^{m_i}$ $m_i \ge 1$, $m_i \le p$ $i = 1, \dots, s$
CILJ: $m_i = p_i$ $i = 1, \dots, s$ $\quad f(x) := (x - \lambda_1)^{m_1} \cdots (x - \lambda_s)^{m_s} \rightarrow f(A) = (A - \lambda_1)^{m_1} \cdots (A - \lambda_s)^{m_s} \overset{*}{=} 0 \xrightarrow{\forall i} \mu_A | f \rightarrow m_i = p_i \rightarrow \mu_{A_i}(x) = (x - \lambda_i)^{m_i} = (x - \lambda_i)^{p_i}$ ★

12 Tm-a: $S := S_1 \dotplus \cdots \dotplus S_s = \overbrace{\lambda_1 I_1}^{\text{skalarni op.}} \dotplus \cdots \dotplus \lambda_s I_s$

$x = x_1 + \cdots + x_s$ za jednstvn. $x_i \in V_i$

$\rightarrow Sx = S_1x_1 + \cdots + S_sx_s$ (β2)

$Sx = \lambda_1 I_1 x_1 + \cdots + \lambda_s I_s x_s$

$\rightarrow$ pokazuje da je op. S poluprost, tj. $\exists$ baza u kojoj je dijaglzb.

Baza za V = $(\underset{\text{bilo koja}}{\text{baza } V_1}, \ldots, \underset{\text{bilo koja}}{\text{baza } V_s})$

$$S_{(baza V)} = \begin{bmatrix} S_{1(baza V_1)} & & 0 \\ & S_{2(baza V_2)} & \\ 0 & \ddots & S_{s(baza V_s)} \end{bmatrix} = \begin{bmatrix} \overbrace{\lambda_1 \ldots \lambda_1}^{\dim V_1} & & 0 \\ & \ddots & \\ 0 & & \overbrace{\lambda_s \ldots \lambda_s}^{\dim V_s} \end{bmatrix}$$

$\forall_{i=1,\ldots,s} \quad (A_i - \underbrace{\lambda_i I_i}_{S_i})^{p_i} = 0 \quad (A_i - \lambda_i I_i)^{p_i - 1} \neq 0 \quad (Tm (e))$

Operator $N_i := A_i - \overbrace{\lambda_i I_i} \in \mathcal{L}(V_i)$ zadovoljava $N_i^{p_i} = 0$, $N_i^{p_i - 1} \neq 0$ $\rightarrow$ N nilpot. ind $= p_i$

Uočimo $S_i = \lambda_i I_i$ skalarni op $\rightarrow$ $S_i N_i = N_i S_i$ komutiraju

$N_i = A_i - \lambda_i I = A_i - S_i \rightarrow A_i = S_i + N_i$

Na kraju:

$N \in \mathcal{L}(V)$ $\quad N := N_1 \dotplus \cdots \dotplus N_s \quad \overset{\text{za } k \geq 1}{} \overset{\beta 3}{\rightarrow} N^k = N_1^{k_1} \dotplus \cdots \dotplus N_s^{k_s} \rightarrow N$ nilpot. ind $= p = \max\{p_1, \ldots, p_s\}$

$\boxed{\begin{matrix} NS = SN \\ A = N + S = S + N \end{matrix}}$ "Jordanova dekompozicija" operatora na sumu polupr. (S) i nilpot. (N) koji komutiraju

Nap. Nema jedinstvenosti ako S' i N' ne komutiraju.

TM. **Jordanova dekompozicija je jedinstvena.**

($A = S' + N'$, S' polupr, N' nilpot. $S'N' = N'S' \rightarrow S' = S$, $N' = N$)

7. GEOMETRIJA UNITARNIH PROSTORA

V (kon. dim.) v.p. nad $K=\mathbb{R}$ ili $\mathbb{C}$. Iz v.p. dobivamo unitarni prostor uvođenjem skalarnog produkta.

Def. **SKALARNI PRODUKT** je f-ja $\langle | \rangle : V\times V \to K$ koja zadovoljava:

a) $\langle v|v\rangle \geq 0$, $\langle v|v\rangle = 0 \iff v=0$ $\quad \forall v\in V$ — STROGA DEFIN.

b) $\langle \alpha v+\beta w|u\rangle = \alpha\langle v|u\rangle + \beta\langle w|u\rangle$ $\quad \forall v,w,u\in V \ \forall \alpha,\beta\in K$ — LIN U PRVOM ARG. (hom + ad.)

c) $\langle v|w\rangle = \overline{\langle w|v\rangle}$ $\quad \forall v,w\in V$ — HERMITSKA SIMETRIČNOST

$(V, \langle | \rangle)$ **UNITARNI PROSTOR** nad poljem K.

- $\langle v|w\rangle = \overline{\langle w|v\rangle}$ nam daje kontrolu drugog argumenta:

  $\langle u|\alpha v+\beta w\rangle = \overline{\langle \alpha v+\beta w|u\rangle} = \overline{\alpha\langle v|u\rangle + \beta\langle w|u\rangle} = \bar\alpha\,\overline{\langle v|u\rangle} + \bar\beta\,\overline{\langle w|u\rangle} = \bar\alpha\langle u|v\rangle + \bar\beta\langle u|w\rangle$ — ANTI-LINEARNOST

  Ako $K=\mathbb{R} \Rightarrow$ dobimo linearnost po drugom argumentu (jer $\bar\alpha=\alpha$; $\bar\beta=\beta$)

- $K=\mathbb{R} \Rightarrow$ c) glasi $\langle v|w\rangle = \langle w|v\rangle$

Primjer:

$V=K^n = \{x=(x_1,\dots,x_n);\ x_i\in K\}$ n-dim. v.p. nad K (baza $(e_1,\dots,e_n)$ - kanonska) — DZ - (e) jest baza

Na tom V staviti ćemo (neke od skalarnih produkata):

- $\forall\, \underline{\lambda} := (\lambda_1,\dots,\lambda_n)\in\mathbb{R}^n \quad \lambda_i>0 \qquad \langle x,y\rangle_{\underline{\lambda}} := \sum_{i=1}^{n}\lambda_i x_i \overline{y_i}$ — DZ: $\langle x|y\rangle_{\underline{\lambda}}$ jest sk. pr.
- Ako $\underline{\lambda}=(1,\dots,1)$ onda dobijemo **kanonski sk. pr.**: $\langle x|y\rangle := \langle x|y\rangle_{\underline{\lambda}} = \sum_{i=1}^{n} x_i\overline{y_i}$

DZ: Općenito, kada $\underline{\lambda}$ varira, $\langle | \rangle_{\underline{\lambda}}$ su bitno drugačiji. Pokažite: $\langle e_i|e_i\rangle_{\underline{\lambda}} = \lambda_i \quad 1\leq i\leq n$

- Mogu se zadati još svakakvi sk. pr.

Zbog svojstva a) možemo definirati **NORMU (DULJINU)** vektora $v\in V$ kao $\|v\| := \sqrt{\langle v|v\rangle}$.
Dobivamo funkciju $\|\cdot\| : V\to\mathbb{R}$ $\quad (\mathrm{Im}(\|\cdot\|) \subseteq [0,+\infty\rangle)$

Def. Vektor $v\in V$ je **normiran** ako $\|v\|=1$.

Prop. Norma $\|\cdot\|$ zadovoljava:

a) $\|v\|\geq 0 \quad \forall v\in V$

b) $\|v\|=0 \iff v=0$

c) $\|\lambda v\| = |\lambda|\,\|v\| \quad \forall\lambda\in K\ \forall v\in V$

d) $\|v+w\| \leq \|v\|+\|w\| \quad \forall v,w\in V$ — NEJEDN. TROK. - posljedica od CSB

DOKAZ:

a) b) $\|v\| = \sqrt{\underbrace{\langle v|v\rangle}_{\geq 0}} \geq 0 \qquad \|v\|=0 = \sqrt{\langle v|v\rangle}=0 \iff \langle v|v\rangle = 0 \iff v=0$ //

c) $\|\lambda v\|^2 = \langle \lambda v|\lambda v\rangle = \lambda\bar\lambda\langle v|v\rangle = |\lambda|^2\|v\|^2 \to \|\lambda v\| = |\lambda|\,\|v\|$ //

d) $\|v+w\|^2 = \langle v+w|v+w\rangle = \langle v|v\rangle + \langle v|w\rangle + \langle w|v\rangle + \langle w|w\rangle = \|v\|^2 + \langle v|w\rangle + \overline{\langle v|w\rangle} + \|w\|^2 =$
$= \|v\|^2 + 2\mathrm{Re}\langle v|w\rangle + \|w\|^2 \leq \|v\|^2 + 2|\langle v|w\rangle| + \|w\|^2 \overset{CSB}{\leq} \|v\|^2 + 2\|v\|\|w\| + \|w\|^2 = (\|v\|+\|w\|)^2$

$\to \|v+w\| \leq \|v\|+\|w\|$ //

(DZ) **Lema 7.3.** Za sve $x,y\in V$ vrijedi:

a) $2(\|x\|^2+\|y\|^2) = \|x+y\|^2 + \|x-y\|^2$ — RELACIJA PARAL.

b) $K=\mathbb{R} \to \langle x|y\rangle = \dfrac{\|x+y\|^2-\|x-y\|^2}{4}$

c) $K=\mathbb{C} \to \langle x|y\rangle = \frac{1}{4}(\|x+y\|^2-\|x-y\|^2) + \frac{i}{4}(\|x+iy\|^2-\|x-iy\|^2)$

DOKAZ:

1) $\|x+y\|^2 = \langle x+y|x+y\rangle = \|x\|^2 + \langle x|y\rangle + \langle y|x\rangle + \|y\|^2$

2) $\|x-y\|^2 = \langle x-y|x-y\rangle = \|x\|^2 - \langle x|y\rangle - \langle y|x\rangle + \|y\|^2$

3) $\|x+iy\|^2 = \langle x+iy|x+iy\rangle = \|x\|^2 - i\langle x|y\rangle + i\langle y|x\rangle + \|y\|^2$

4) $\|x-iy\|^2 = \langle x-iy|x-iy\rangle = \|x\|^2 + i\langle x|y\rangle - i\langle y|x\rangle + \|y\|^2$

a) $1)+2) \to \|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2+\|y\|^2)$ //

b) $K=\mathbb{R} \to \langle x|y\rangle = \overline{\langle y|x\rangle} = \langle y|x\rangle \to 1)-2) \to \|x+y\|^2 - \|x-y\|^2 = 4\langle x|y\rangle$ //

c) $1)-2) \to \|x+y\|^2 - \|x-y\|^2 = 2(\langle x|y\rangle + \langle y|x\rangle)$
$3)i - 4)i \to i(\|x+iy\|^2 - \|x-iy\|^2) = 2(\langle x|y\rangle - \langle y|x\rangle)$ $\quad + \to 4\langle x|y\rangle = \|x+y\|^2 - \|x-y\|^2 + i(\|x+iy\|^2 - \|x-iy\|^2)$ //

CAUCHY-SCHWARZ-BUNJAKOVSKI

Prop. Neka je V unit. pr. $\forall x,y \in V$ vrijedi $|\langle x|y\rangle| \leq \|x\| \cdot \|y\|$.

DOKAZ:

$v_t := ty + x$ (t prolazi po $K = \mathbb{R}$ ili $\mathbb{C}$)

$0 \leq \|v_t\|^2 = \langle v_t|v_t\rangle = \langle ty+x \mid ty+x\rangle = \langle ty|ty\rangle + \langle ty|x\rangle + \langle x|ty\rangle + \langle x|x\rangle = t\bar{t}\langle y|y\rangle + t\langle y|x\rangle + \bar{t}\langle x|y\rangle + \langle x|x\rangle \to *$

$t := -\frac{\langle x|y\rangle}{\langle y|y\rangle}$ ($y \neq 0$, za $y=0 \to 0=0 \checkmark$)

$* \to -\frac{\langle x|y\rangle}{\langle y|y\rangle} \cdot \frac{-\overline{\langle x|y\rangle}}{\overline{\langle y|y\rangle}} \cdot \langle y|y\rangle + \frac{-\langle x|y\rangle}{\langle y|y\rangle}\langle y|x\rangle + \frac{-\overline{\langle x|y\rangle}}{\langle y|y\rangle}\langle x|y\rangle + \langle x|x\rangle \geq 0 \quad / \cdot \langle y|y\rangle$ $\quad$ $\langle y|y\rangle \in \mathbb{R}$ jer $\langle y|y\rangle \geq 0$

$\cancel{\langle x|y\rangle\overline{\langle x|y\rangle}} \underbrace{- \langle x|y\rangle \overset{\overline{\langle x|y\rangle}}{\overline{\langle y|x\rangle}}}_{-|\langle x|y\rangle|^2} - \cancel{\overline{\langle x|y\rangle}\langle x|y\rangle} + \underbrace{\langle x|x\rangle}_{\|x\|^2}\underbrace{\langle y|y\rangle}_{\|y\|^2} \geq 0 \longrightarrow |\langle x|y\rangle| \leq \|x\|\cdot\|y\|$

Jednakost vrijedi u slučaju lin. zav. vektora $x,y \in V$. (Preskačemo taj dio dokaza)

Def.
$x,y \in V$ $\quad$ x **OKOMIT NA** y: $\quad$ $x \perp y$ ako $\langle x|y\rangle = 0$
$x \in V,\ W \leq V$ $\quad$ x **OKOMIT NA** W: $\quad$ $x \perp W$ ako $x \perp y\ \forall y \in W$
$W_1, W_2 \leq V$ $\quad$ W_1 **OKOMIT NA** W_2: $\quad$ $W_1 \perp W_2$ ako $x \perp W_2\ \forall x \in W_1$ akko $x \perp y\ \forall x \in W_1\ \forall y \in W_2$

DZ.
ISK + D
VAŽNO!

Prop. 7.5. i) Pitagorin poučak: $x \perp y \to \|x+y\|^2 = \|x\|^2 + \|y\|^2$
ii) $x \perp y_1, y_2, \dots, y_m \to x \perp [\{y_1, \dots, y_m\}]$
iii) $x \perp V \to x = 0$

DOKAZ:

i) $\|x+y\|^2 = \langle x+y|x+y\rangle = \langle x|x\rangle + \overset{0}{\widehat{\langle x|y\rangle}} + \overset{0}{\widehat{\langle y|x\rangle}} + \langle y|y\rangle = \|x\|^2 + \|y\|^2 \checkmark$

ii) $v \in [\{y_1, \dots, y_m\}] \leftrightarrow \exists \alpha_1, \dots, \alpha_m \in K$ t.d. $v = \sum_{i=1}^{m} \alpha_i y_i$ $\quad / \langle x| \ \rangle$

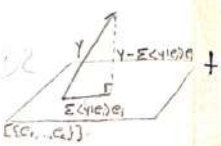
$\to \langle x|v\rangle = \sum_{i=1}^{m} \overline{\alpha_i}\overset{0}{\widehat{\langle x|y_i\rangle}} = 0 \to x \perp v \checkmark$

iii) $x \perp V \to x \perp x \to \langle x|x\rangle = 0 \to x = 0$

Def. Niz vektora $e_1, \dots, e_k$ unitarnog prostora V je **ortonormiran** ako $e_i \perp e_j\ \forall i \neq j \wedge \|e_i\| = 1\ \forall i = 1, \dots, k$

DZ
VAŽNO

Prop. 7.6. Neka je $e_1, \dots, e_k$ ON skup vektora unitarnog prostora V.
Tada: 1) $e_1, \dots, e_k$ su lin. nez.
2) $k \leq n = \dim V$
3) $\dim [\{e_1, \dots, e_k\}] = k \leq n$



te nadalje, za svaki $x \in [\{e_1, \dots, e_k\}]$ vrijedi $x = \sum_{i=1}^{k} \langle x|e_i\rangle e_i$ (prikaz u bazi $(e_1, \dots, e_k)$ ptp. $[\{e_1, \dots, e_k\}]$)

4) $y \in V \to y - \boxed{\sum_{i=1}^{k} \langle y|e_i\rangle e_i} \perp [\{e_1, \dots, e_k\}]$

$\hookrightarrow$ naziva se ORTOGONALNA PROJEKCIJA vektora y na ptp. $[\{e_1, \dots, e_k\}]$

VAŽNO! Svostvo ortogonalnosti potprostora unitarnog prostora V

Prop. Def. Neka su $W_1, \dots, W_\ell$ ptp. unit. pr. V t.d. $W_i \perp W_j\ \forall i \neq j$. Tada je suma $W_1 + \dots + W_\ell$ direktna i naziva se ORTOGONALNA SUMA potprostora $W_1, \dots, W_\ell$ i ima posebnu oznaku $W_1 \oplus \dots \oplus W_\ell$.

DOKAZ:

Iz poček. §3: treba pokazati: $(w_1 + \dots + w_\ell = 0_V,\ w_i \in W_i) \to (w_1 = \dots = w_\ell = 0_V)$

$w_1 + w_2 + \dots + w_\ell = 0 \ / \langle\ |w_1\rangle \quad \to \quad \langle w_1|w_1\rangle + \underbrace{\langle w_2|w_1\rangle}_{=0\ (W_2 \perp W_1)} + \dots + \underbrace{\langle w_\ell|w_1\rangle}_{=0\ (W_\ell \perp W_1)} = 0 \quad \to \langle w_1|w_1\rangle = 0 \to w_1 = 0$

Analogno: $w_1 = \dots = w_\ell = 0 \Rightarrow$ suma direktna

V unit. pr., $\langle\cdot|\cdot\rangle$ sk. pr.

Def. Neka je $x_1, x_2, \dots, x_k \in V$. Definiramo:

a) GRAMOVU MATRICU := $G(x_1,\dots,x_k) := \begin{bmatrix} \langle x_1|x_1\rangle & \cdots & \langle x_1|x_k\rangle \\ \vdots & & \vdots \\ \langle x_k|x_1\rangle & \cdots & \langle x_k|x_k\rangle \end{bmatrix}_{k\times k}$ → $k\times k$ matrica sa koef. $\in \mathbb{K}$ ($\mathbb{R}$ ili $\mathbb{C}$)

b) GRAMOVU DETERMINANTU: $\Gamma(x_1,\dots,x_k) := \det G(x_1,\dots,x_k)$

TEHNIČKA LEMA

Za $\underset{\alpha_1,\dots,\alpha_k \in \mathbb{K}}{\overset{x_1,\dots,x_k\in V}{}}$ vrijedi $\sum_{i=1}^{k}\alpha_i x_i = 0$ akko $\overset{k\text{-torka}}{(\alpha_1,\dots,\alpha_k)}$ je rješenje $\overset{(*)}{\text{kvadr. hom. sustava}}$

$$(*)\begin{cases} \sum_{i=1}^{k}\alpha_i\langle x_i|x_1\rangle = 0 \\ \sum_{i=1}^{k}\alpha_i\langle x_i|x_2\rangle = 0 \\ \vdots \\ \sum_{i=1}^{k}\alpha_i\langle x_i|x_k\rangle = 0 \end{cases} \equiv \left[G(x_1,\dots,x_k)\right]^t \begin{bmatrix}\alpha_1\\ \vdots \\ \alpha_k\end{bmatrix} = \begin{bmatrix}0\\ \vdots \\ 0\end{bmatrix}$$

ovo je mat. zapis od $(*)$

DOKAZ:

$\Rightarrow$ $\sum \alpha_i x_i = 0 \;/\cdot\langle\;|x_j\rangle$ $j=1,\dots,k$; dobimo hom. sust. jer je sk. pr. lin. po 1. arg

$\Leftarrow$ $y := \sum_{i=1}^{k}\alpha_i x_i$ tada iz hom. sustava nalazimo $\langle y|x_1\rangle = \sum_{i=1}^{k}\alpha_i\langle x_i|x_1\rangle = 0$ itd. $\langle y|x_k\rangle = 0$

$\langle y|x_1\rangle = 0 \;/\cdot\overline{\alpha_1} \;\to\; \overline{\alpha_1}\langle y|x_1\rangle = 0 \;\to\; \langle y|\alpha_1 x_1\rangle = 0$ itd $\langle y|\alpha_k x_k\rangle = 0$

Sve ih zbrojimo: $0 = 0+\dots+0 = \langle y|\alpha_1 x_1\rangle + \dots + \langle y|\alpha_k x_k\rangle = \langle y|\sum_{i=1}^{k}\alpha_i x_i\rangle = \langle y|y\rangle \to y = 0 \to \sum_{i=1}^{k}\alpha_i x_i = 0$

KOROLAR TEHNIČKE LEME

Tm. $x_1, x_2, \dots, x_k$ su lin. nez. akko $\Gamma(x_1,\dots,x_k) \neq 0$.

DOKAZ:

$x_1,\dots,x_k$ lin. nez. $\leftrightarrow$ $\left(\sum_{i=1}^{k}\alpha_i x_i = 0 \to \alpha_1 = \dots = \alpha_k = 0\right) \overset{\text{TEH.}}{\underset{\text{LEMA}}{\leftrightarrow}}$ odgovarajući hom. sustav ima jedinstveno rj. $\leftrightarrow$

$\overset{\text{mat.}}{\underset{\text{zapis}}{\leftrightarrow}} \underbrace{\det[G(x_1,\dots,x_k)]^t}_{=\det[G(x_1,\dots,x_k)] = \Gamma(x_1,\dots,x_k)} \neq 0$

DZ Lema 7.10.

ISKAZ

Neka je $x_1,\dots,x_m$ niz lin. nez. vektora u V. Pretp. da niz ort. norm. vektora $e_1,\dots,e_m$ zadovoljava

i); ii) $[\{x_1,\dots,x_k\}] = [\{e_1,\dots,e_k\}]$ $\forall k=1,\dots,m$ i $\langle e_k|x_k\rangle > 0$ $\forall k=1,2,\dots,m$. Tada za svaki $2 \leq k \leq m$ i $e\in V$ t.d. $\|e\|=1$, $e\neq e_k$ i $e \perp [\{e_1, e_2,\dots,e_{k-1}\}]$ vrijedi $\|e - x_k\| > \|e_k - x_k\|$.

Nadalje, postoji najviše jedan niz ortonom. vektora $e_1,\dots,e_m$ t.d. vrijedi (i) i (ii).

EKSPLICITAN POSTUPAK GRAM-SCHMITOVE ORTOGONALIZACIJE – temelji se na prethodnom tm.-u

- polazimo od lin. nez. niza vektora $x_1, x_2,\dots,x_m$ u V, to nam odmah daje: $m \leq \dim V = n$
- uočimo: $\forall k$ $1\leq k\leq m$ slijedi $x_1,\dots,x_k$ lin. nez. te prema prethodnom tm.-u imamo: $\Gamma(x_1,\dots,x_k) \neq 0$

(Zašto? Jer ako bi bilo $\sum_{i=1}^{k}\alpha_i x_i = 0$, onda proširimo $\sum_{i=1}^{k}\alpha_i x_i + 0x_{k+1} + \dots + 0x_m = 0$, a sada zbog lin. nez. skupa $x_1,\dots,x_m$ mora biti $\alpha_1 = \dots = \alpha_k = 0$.)

Definiramo novi niz vektora:

$$y_1 := x_1$$

$$y_2 := \begin{vmatrix} \langle x_1|x_1\rangle & x_1 \\ \langle x_2|x_1\rangle & x_2 \end{vmatrix} \overset{\text{Lap. razvoj po zadnjem st.}}{:=} \underbrace{\langle x_1|x_1\rangle}_{\Gamma(x_1)} x_2 - \underbrace{\langle x_2|x_1\rangle x_1}_{\in[\{x_1\}]}$$

$$y_3 := \begin{vmatrix} \langle x_1|x_1\rangle & \langle x_1|x_2\rangle & x_1 \\ \langle x_2|x_1\rangle & \langle x_2|x_2\rangle & x_2 \\ \langle x_3|x_1\rangle & \langle x_3|x_2\rangle & x_3 \end{vmatrix} \overset{\text{Lap. razvoj po zadnjem st.}}{=} \underbrace{x_1\begin{vmatrix} \langle x_2|x_1\rangle & \langle x_2|x_2\rangle \\ \langle x_3|x_1\rangle & \langle x_3|x_2\rangle \end{vmatrix} - x_2\begin{vmatrix} \langle x_1|x_1\rangle & \langle x_1|x_2\rangle \\ \langle x_3|x_1\rangle & \langle x_3|x_2\rangle \end{vmatrix}}_{\in[\{x_1,x_2\}]} + x_3\underbrace{\begin{vmatrix} \langle x_1|x_1\rangle & \langle x_1|x_2\rangle \\ \langle x_2|x_1\rangle & \langle x_2|x_2\rangle \end{vmatrix}}_{\Gamma(x_1,x_2)}$$

Općenito $2\leq k\leq m$:

$$y_k := \begin{vmatrix} \langle x_1|x_1\rangle & \langle x_1|x_2\rangle & \cdots & \langle x_1|x_{k-1}\rangle & x_1 \\ \langle x_2|x_1\rangle & \langle x_2|x_2\rangle & \cdots & \langle x_2|x_{k-1}\rangle & x_2 \\ \vdots & \vdots & & \vdots & \vdots \\ \langle x_{k-1}|x_1\rangle & \langle x_{k-1}|x_2\rangle & \cdots & \langle x_{k-1}|x_{k-1}\rangle & x_{k-1} \\ \langle x_k|x_1\rangle & \langle x_k|x_2\rangle & \cdots & \langle x_k|x_{k-1}\rangle & x_k \end{vmatrix} := \text{Lap. razvoj po zadnjem stupcu} =$$

$$= (-1)^{k+k}\,\Gamma(x_1,\dots,x_{k-1})\,x_k + \text{nešto iz } [\{x_1,\dots,x_{k-1}\}]$$

$$= \underbrace{\Gamma(x_1,\dots,x_{k-1})}_{\neq 0 \text{ jer su } x_1,\dots,x_{k-1} \text{ lin. nez}} x_k + \text{nešto iz } [\{x_1,\dots,x_{k-1}\}] \quad \text{SUPER VAŽNO!}$$

TM.
1) $y_1,\dots,y_m$ su lin. nez
2) $\forall k$ $2\leq k\leq m$ vrijedi $y_k \perp [\{y_1,\dots,y_{k-1}\}]$ ($\leftrightarrow$ $y_k \perp y_1,\dots,y_{k-1}$)
3) $y_i \perp y_j$ $i\neq j$
4) $e_i := \frac{y_i}{\|y_i\|}$ (ovo možemo definirati jer zbog 1) je $y_i \neq 0_V$). Tada je $e_1,\dots,e_n$ ON niz vektora koji zadovoljava:
   A) $[\{x_1,\dots,x_k\}] = [\{e_1,\dots,e_k\}]$ $\forall k=1,\dots,m$
   B) $\langle e_k|x_k\rangle > 0$ $\forall k=1,\dots,m$
   → Lema 7.10. garantira jedinstvenost niza s ta dva svojstva

DOKAZ – imam u bilj

KOR. $x_1, x_2,\dots,x_k$ lin. nez. $\longrightarrow$ $\Gamma(x_1,x_2,\dots,x_k) \in \mathbb{R}$ i $\Gamma(x_1,\dots,x_k) > 0$.

DZ Iskazi i dokazi:
BITNA

Kor 7.23. V kon.dim.unit.v.p. Tada $\exists$ ONB za V.
Nadalje, $(e)=(e_1,\dots,e_n)$ ONB za $V \rightarrow \forall x\in V$ vrijedi $x=\sum_{i=1}^{n}\langle x|e_i\rangle e_i$

DOKAZ:
- Neka je $(x_1,\dots,x_n)$ baza za V. Tada G-S postupkom dobivamo ON niz vektora $e_1,\dots,e_n$ $((e):=(e_1,\dots,e_n))$
Oni su lin.nez. Slijedi (e) je ONB za V.
- Prikaz $x=\sum_{i=1}^{n}\langle x|e_i\rangle e_i$ slijedi iz Fourierovog razvoja

TM. 7.24. (tm. o projekciji)

Neka je W ptp. up. V. Tada $\forall v\in V$ $\exists! w\in W$ t.d. $v-w\perp W$.
(Drugim riječima, $V=W\oplus W^{\perp}$.) w zovemo ORTOGONALNOM PROJEKCIJOM vektora v na ptp. W

DOKAZ:
- $W=\{0\}$ $\longrightarrow$ Jasno iz def. ort.kompl. $W^{\perp}=V$
- $W\neq\{0\}$: Izaberemo ONB $(e):=(e_1,\dots,e_m)$ za W. Stavimo li $w=\sum_{i=1}^{m}\langle v|e_i\rangle e_i \xrightarrow{\text{prop. 7.6. ii)}} v-w\perp W$

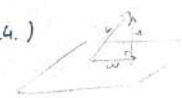
Jedinstvenost: $\underbrace{w,w'\in W}$ t.d. $v-w\perp W \wedge v-w'\perp W \rightarrow v-w\in W^{\perp} \wedge v-w'\in W^{\perp} \rightarrow$
$(v-w)-(v-w')\in W^{\perp} \rightarrow \left.\begin{array}{l} w'-w\in W^{\perp} \\ w'-w\in W \end{array}\right] \xrightarrow[W\cap W^{\perp}=\{0\}]{W\oplus W^{\perp}} \boxed{w=w'}$

KOR 7.25 $\dim W^{\perp}=\dim V-\dim W=n-\dim W$

DOKAZ:
Neka je $W\neq\{0\}$
- $(x_1,\dots,x_m)$ baza za W, $(x_1,\dots,x_{m-1},x_m,x_{m+1},\dots,x_n)$ baza za $V \xrightarrow{GS} (e)=(e_1,\dots,e_n)$ ONB za V
$\rightarrow e_1,\dots,e_m\in W$ i lin.nez. su $\rightarrow (e_1,\dots,e_m)$ ONB za W
- $x\in V \rightarrow x=\sum_{i=1}^{n}\langle x|e_i\rangle e_i$

$(x\in W^{\perp} \leftrightarrow \langle x|e_i\rangle=0 \quad 1\leq i\leq m) \rightarrow (x\in W^{\perp} \leftrightarrow x=\sum_{i=m+1}^{n}\langle x|e_i\rangle e_i) \rightarrow (e_{m+1},\dots,e_n)$ ONB za $W^{\perp} \rightarrow \dim W^{\perp}=n-m$

$W\leq V \rightarrow \forall v\in V$ $\exists!$ prikaz $v=w+u$ $w\in W$, $u\in W^{\perp}$ (Iz tm. 7.24.)



TM. 7.26. (tm. o najboljoj aproksimaciji)

Uz gornje oznake $\|v-w\|<\|v-w'\|$, $\forall w'\in W$, $w'\neq w$

DOKAZ
Po def. ort. pr. imamo $v-w\perp W$. $\rightarrow v-w\in W^{\perp}$
$w-w'\in W \rightarrow \underbrace{w-w'}_{\in W}\perp\underbrace{v-w}_{\in W^{\perp}} \xrightarrow{\text{Pit.}} \|v-w'\|^2=\|\underbrace{(v-w)}_{\in W^{\perp}}+\underbrace{(w-w')}_{\in W}\|^2=\|v-w\|^2+\underbrace{\|w-w'\|^2}_{>0 \text{ za } w\neq w'}>\|v-w\|^2$
(Jer za $w'\neq w$ imamo $\|w-w'\|>0$.)

8. REPREZENTACIJA LINEARNIH FUNKCIONALA I HERMITSKO ADJUNGIRANJE

V bilo koji v.p. nad $\mathbb{K}$ ($\mathbb{K}=\mathbb{C}$ ili $\mathbb{R}$)

a) ponovimo u malo drugačijim oznakama: $\varphi: V \to W$ lin. op. ako $\varphi(\alpha x+\beta y)=\alpha\varphi(x)+\beta\varphi(y)$ $\forall \alpha,\beta \in \mathbb{K}$ $\forall x,y\in V$

b) novi pojam: **ANTILINEARAN OPERATOR** (preslikavanje): $\psi: V\to W$ $\psi(\alpha x+\beta y)=\bar{\alpha}\psi(x)+\beta\psi(y)$, $\forall \alpha,\beta\in\mathbb{K}$ $\forall x,y\in V$

$\mathbb{K}=\mathbb{R}$ $\alpha,\beta\in\mathbb{K} \to \alpha=\bar{\alpha} \wedge \beta=\bar{\beta}$ te zato svaki antilin. op. je u stvari lin. op (tj. pojam b) ne postoji)

$\mathbb{K}=\mathbb{C}$ $\alpha,\beta\in\mathbb{C} \to$ općenito $\alpha\neq\bar{\alpha} \wedge \beta\neq\bar{\beta}$ te se radi o novom pojmu, ali nije "previše novi"

$\Updownarrow$ $\Updownarrow$

$\alpha\in\mathbb{C}\setminus\mathbb{R}$ $\beta\in\mathbb{C}\setminus\mathbb{R}$

$$\psi(\underbrace{\alpha x+\beta y}) = \underbrace{\bar{\alpha}\,\psi(x)+\bar{\beta}\,\psi(y)}$$

množenje vektora $x,y\in V$ skalarima $\alpha,\beta\in\mathbb{K}$ pa zbrajanje u V

vektori $\psi(x)\in W$, $\psi(y)\in W$ množe se skalarima $\bar{\alpha}$ i $\bar{\beta}$, i zatim zbrajaju u W

W je v.p. što uključuje:

① zbrajanje koje je dobro definirano sa svojstvima analognim zbrajanju u $\mathbb{K}$ (Abelova gr.)

② množenje skalarom:

$1w=w$ $\forall w\in W$

$(\gamma+\delta)w=\gamma w+\delta w$ $\forall\gamma,\delta\in\mathbb{K}$, $\forall w\in W$

$\gamma(w_1+w_2)=\gamma w_1+\gamma w_2$ $\forall\gamma\in\mathbb{K}$ $\forall w_1,w_2\in W$

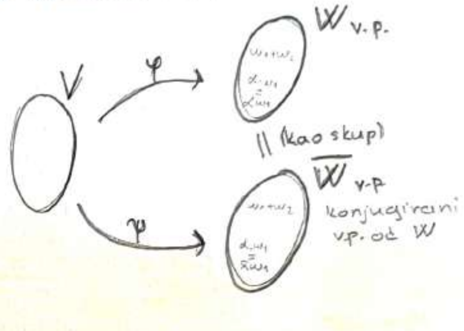
$\gamma(\delta w)=(\gamma\delta)w$ $\forall\gamma,\delta\in\mathbb{K}$ $\forall w\in W$

Konstruiramo novi v.p. $\overline{W}$ kojeg nazivamo **KONJUGIRANI V.P. OD W** (tj. OD PROSTORA W)

③ Kao skup $W=\overline{W}$

④ Zbrajanje u $\overline{W}$ isto kao u W (i zadovoljava analogon od ①)

⑤ Novo množenje skalarom $\alpha . w := \underbrace{\bar{\alpha}w}_{\text{stara množenje}}$ $\forall\alpha\in\mathbb{K}$ $\forall w\in\overline{W}=W$



DZ $\overline{W}$ je v.p. (treba provjeriti analogon od ② za novo mn. sk.)

$1.w=\bar{1}w=1w$ $\forall w\in\overline{W}$

$(\gamma+\delta).w=\overline{(\gamma+\delta)}w=\bar{\gamma}w+\bar{\delta}w=\gamma.w+\delta.w$ $\forall\gamma,\delta\in\mathbb{K}$ $\forall w\in\overline{W}$

$\gamma.(w_1+w_2)=\bar{\gamma}(w_1+w_2)=\bar{\gamma}w_1+\bar{\gamma}w_2=\gamma.w_1+\gamma.w_2$ $\forall\gamma\in\mathbb{K}$ $\forall w_1,w_2\in\overline{W}$

$\gamma.(\delta.w)=\bar{\gamma}(\delta.w)=\bar{\gamma}(\bar{\delta}w)=\overline{\gamma\delta}w=\gamma\delta.w$ $\forall\gamma,\delta\in\mathbb{K}$ $\forall w\in\overline{W}$

DZ $(f)=(f_1,\dots,f_m)$ baza za W $\longleftrightarrow$ (f) baza za $\overline{W}$ te zato $\dim\overline{W}=\dim W$

$\Rightarrow$ $\alpha_1 f_1+\dots+\alpha_m f_m=0_V \to \alpha_1=\dots=\alpha_m=0$

$\alpha_1.f_1+\dots+\alpha_m.f_m=0_V \leftrightarrow \bar{\alpha}_1 f_1+\dots+\bar{\alpha}_m f_m=0_V \xrightarrow{(f)\text{ baza za } W} \bar{\alpha}_1=\dots=\bar{\alpha}_m=0 \to \alpha_1=\dots=\alpha_m=0 \to (f)$ baza za $\overline{W}$

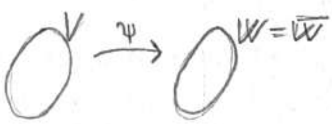
$\Leftarrow$ $\alpha_1.f_1+\dots+\alpha_m.f_m=0_V \to \alpha_1=\dots=\alpha_m=0$

$\alpha_1.f_1+\dots+\alpha_m.f_m=\bar{\alpha}_1 f_1+\dots+\bar{\alpha}_m f_m=0_V=0\cdot f_1+\dots+0\cdot f_m=0_V \to (f)$ baza za W

Što je antilinearan operator?

$$\psi(\underbrace{\alpha x+\beta y}_{\in V})=\bar{\alpha}\underbrace{\psi(x)}_{\in W}+\bar{\beta}\underbrace{\psi(y)}_{\in W}=\alpha.\underbrace{\psi(x)}_{\in\overline{W}}+\beta.\underbrace{\psi(y)}_{\in\overline{W}}$$

te vidimo da je svako antilin.pr. $\psi: V\to W$ zapravo linearno pr. (op.) $\psi: V\to\overline{W}$, tj. $\psi\in\mathcal{L}(V,\overline{W})$.



Zato antilin.op. zadovoljavaju uobičajena svojstva lin.op. poput tm.o rangu i defektu ili ako su npr. $\dim V=\dim W$ onda ψ antilin. izomorfizam (bijekcija) $\leftrightarrow$ $\ker\psi=\{0\}$, tj. $\psi(v)=0 \to v=0$

Jedna primjena ovih pojmova i rezultata:

Tm. (O REPREZENTACIJI LIN. FUNKCIONALA)

Neka je V unitaran prostor sa sk.pr. $\langle\cdot|\cdot\rangle$. Tada: a) $\forall y\in V$ presl. $x\mapsto\langle x|y\rangle$, $V\to\mathbb{K}$, je lin. funkc. na V, (tj. $\in V^*$)

b) $\forall f\in V^*$ $\exists! y\in V$ t.d. $f(x)=\langle x|y\rangle$ $\forall x\in V$

dokaz

Ponovimo: f lin. funkc na V, tj. $f\in V^*$ akko $f(\alpha_1 x_1+\alpha_2 x_2)=\alpha_1 f(x_1)+\alpha_2 f(x_2)$ $f: V\to\mathbb{K}$ $\forall\alpha_1,\alpha_2\in\mathbb{K}$ $\forall x_1,x_2\in V$

zbrajanje i mn. u $\mathbb{K}$

a) slijedi odmah jer je sk.pr linearan u 1. argumentu $\langle\alpha_1 x_1+\alpha_2 x_2|y\rangle=\alpha_1\langle x_1|y\rangle+\alpha_2\langle x_2|y\rangle$

b) Za $y\in V$ označimo sa $\psi(y)$ lin. funkc $x\mapsto\langle x|y\rangle$ tj. $\psi(y)(x)=\langle x|y\rangle$ $\forall x,y\in V$

Mi možemo tumačiti ψ kao preslikavanje $\psi: V\to V^*$, $V\ni y\mapsto\psi(y)\in V^*$, $\psi(y)(x)=\langle x|y\rangle$ $\forall x\in V$

Kako je sk.pr. antilinearan u 2. argumentu, tj. $\langle x|\beta_1 y_1+\beta_2 y_2\rangle=\bar{\beta}_1\langle x|y_1\rangle+\bar{\beta}_2\langle x|y_2\rangle$ $\forall\beta_1,\beta_2\in\mathbb{K}$ $\forall x,y_1,y_2\in V$

$\psi(\beta_1 y_1+\beta_2 y_2)(x)=\bar{\beta}_1\psi(y_1)(x)+\bar{\beta}_2\psi(y_2)(x)$ $\forall x\in V$

Kraće: $\psi(\beta_1 y_1+\beta_2 y_2)=\bar{\beta}_1\psi(y_1)+\bar{\beta}_2\psi(y_2)$ antilin. pr. $V\to V'$

Koja jezgra?

$\psi(y)=0$ znači $\psi(y)(x)=0$ $\forall x\in V$ znači $\langle x|y\rangle=0$ $\forall x\in V$

Uzmemo $x=y$, nalazimo $\langle y|y\rangle=0 \to y=0 \to \ker\psi=\{0\}$

$(\ker\psi=\{0\} \wedge \dim V=\dim V') \to \psi$ antilin. izom. $\to \forall f\in V^*$ $\exists! y\in V$ t.d. $f=\psi(y)$ (ili drugačije: $f(x)=\psi(y)(x)=\langle x|y\rangle$ $\forall x\in V$)

Primjena tm.-a o reprezentaciji lin. funkcionala:

- V - unitaran prostor $\langle\cdot|\cdot\rangle_V$
- W - unitaran prostor $\langle\cdot|\cdot\rangle_W$
- $A\in\mathcal{L}(V,W)$ - bilo koji lin. op.

$\forall v\in V$ $Av\in W$ te možemo računati sk.pr. $\langle\cdot|\cdot\rangle_W$ s bilo kojim vektorom $w\in W$; $\langle Av|w\rangle_W\in\mathbb{K}$

Dakle, posebno imamo preslikavanje $f: V\to\mathbb{K}$ $v\mapsto\langle Av|w\rangle_W$ (privremeno ga zovemo f)

$f(v)=\langle Av|w\rangle_W$ $\forall v\in V$

$f(\alpha_1v_1+\alpha_2v_2)=\langle A(\alpha_1v_1+\alpha_2v_2)|w\rangle_W=[\text{lin. od } A\in\mathcal{L}(V,W)]=\langle\alpha_1Av_1+\alpha_2Av_2|w\rangle_W=[\text{lin. od. } \langle\cdot|\cdot\rangle_W]\overset{\to\text{ sk.pr. je lin u 1. arg.}}{=}\alpha_1\langle Av_1|w\rangle_W+\alpha_2\langle Av_2|w\rangle_W$

$=\alpha_1f(v_1)+\alpha_2f(v_2)\Rightarrow f$ jest lin. funkc. na V, tj. $f\in V^*$

Nadalje, ako držimo A fiksiranim, f ovisi samo o izboru $w\in W$.

Iduće, tm. o rep. lin. funkc. kaže da mora postojati jedinstveni vektor $y\in V$ t.d. $f(v)=\langle v|y\rangle_V$

DRUGIM RIJEČIMA, $\underbrace{\forall w\in W\ \exists!\, y\in V}_{\text{definira fju: } W\to V,\ w\mapsto y=:A^*w}$ t.d. $\underbrace{\langle Av|w\rangle_W}_{f}=\underbrace{\langle v|y\rangle_V}_{\text{drugi prikaz za } f}$ $\forall v\in V$

$A^*: W\to V$

$A^*w=y$ jedinstveno određen s $\langle Av|w\rangle_W=\langle v|\underbrace{A^*w}_{y}\rangle_V$

Prop $A^*\in\mathcal{L}(W,V)$ (tj. A^* je lin. op.)

DOKAZ.

Treba pokazati $A^*(\beta_1w_1+\beta_2w_2)=\beta_1A^*w_1+\beta_2A^*w_2$ $\forall\beta_1,\beta_2\in\mathbb{K}$, $\forall w_1,w_2\in W$

To ne ide direktno, već uzmemo $v\in V$ proizvoljan i računamo:

$\langle v|A^*(\beta_1w_1+\beta_2w_2)\rangle_V=(\text{def. od } A^*)=\langle Av|\beta_1w_1+\beta_2w_2\rangle_W=\overline{\beta_1}\langle Av|w_1\rangle_W+\overline{\beta_2}\langle Av|w_2\rangle_W=(\text{def. od } A^*)$

$=\overline{\beta_1}\langle v|A^*w_1\rangle_V+\overline{\beta_2}\langle v|A^*w_2\rangle_V=\langle v|\beta_1A^*w_1+\beta_2A^*w_2\rangle_V$

Vidimo da početak i kraj daju reprezentaciju istog lin. funkcionala iz V' s vektorima $A^*(\beta_1w_1+\beta_2w_2)$ i $(\beta_1A^*w_1+\beta_2A^*w_2)$, zbog toga su jednaki, što znači linearnost od A^*.

VAŽNA DZ MATRIČNI ZAPIS HERMITSKOG ADJUNGIRANJA

DZ

TM. 8.4.

KOR. 8.6.

TM. 8.5 - bez dokaza

$A\in\mathcal{L}(V,W)$, $B\in\mathcal{L}(W,U)$

$\overrightarrow{(BA)^*}=A^*B^*$

(tj. $BA\in\mathcal{L}(V,U)$)

matrica A

$A\in M_{mn}(\mathbb{K})$ $\xrightarrow[\text{kompl.-konj.}]{\text{transp.}}$ herm. adj. matrica matrici A

$A=(a_{ij})$

$A^*\in M_{nm}(\mathbb{K})$

$A^*=(\overline{a_{ji}})$

V u.v.p., sk.pr. $\langle\cdot|\cdot\rangle_V$ $(e)=(e_1,\dots,e_n)$ ONB za V

W u.v.p., sk.pr. $\langle\cdot|\cdot\rangle_W$ $(f)=(f_1,\dots,f_m)$ ONB za W

$A\in\mathcal{L}(V,W)$, tada možemo pisati: $\begin{cases}Ae_1=\alpha_{11}f_1+\alpha_{21}f_2+\dots+\alpha_{m1}f_m\\ \vdots\\ Ae_n=\alpha_{1n}f_1+\alpha_{2n}f_2+\dots+\alpha_{mn}f_m\end{cases}$,

na taj način operatoru A pridružujemo matricu:

$\to$ matrica op. A u paru baza f i e

$$A_{(f,e)}=\begin{bmatrix}\alpha_{11}&\alpha_{12}&\dots&\alpha_{1n}\\ \alpha_{21}&\alpha_{22}&\dots&\alpha_{2n}\\ \vdots&\vdots&&\vdots\\ \alpha_{m1}&\alpha_{m2}&\dots&\alpha_{mn}\end{bmatrix}\in M_{mn}(\mathbb{K})$$

Nadalje, jer su (e) i (f) ONB, imamo: $\alpha_{ij}=\alpha_{ij}\langle f_i|f_i\rangle_W=\langle\alpha_{ij}f_i|f_i\rangle_W=\langle\alpha_{1j}f_1+\dots+\alpha_{mj}f_m|f_i\rangle_W$

$=\langle Ae_j|f_i\rangle_W$, $1\le i\le m$, $1\le j\le n$

Kako je $A^*\in\mathcal{L}(W,V)$, to slično kao gore imamo matr. zap. op A^* u paru baza e i f: $A^*f_i=\beta_{1i}e_1+\dots+\beta_{ni}e_n$,

te na taj način op. A^* pridružujemo matricu:

$$A^*_{(e,f)}=\begin{bmatrix}\beta_{11}&\beta_{12}&\dots&\beta_{1m}\\ \beta_{21}&\beta_{22}&\dots&\beta_{2m}\\ \vdots&\vdots&&\\ \beta_{n1}&\beta_{n2}&\dots&\beta_{nm}\end{bmatrix}\in M_{nm}(\mathbb{K})$$

Analogno kao gore, nalazimo $\beta_{ij}=\langle A^*f_j|e_i\rangle_V$ $1\le i\le n$, $1\le j\le m$.

Dalje, nalazimo: $\beta_{ij}=\langle A^*f_j|e_i\rangle_V=\langle f_j|Ae_i\rangle_W=\overline{\langle Ae_i|f_j\rangle_V}=\overline{\alpha_{ji}}$, $1\le i\le n$, $1\le j\le n$. Time je dokazano:

Prop. 8.7. U paru ONB (e) i (f), matrica $(A^*)_{(e,f)}$ operatora A^* je hermitski adjungirana matrici $A_{(f,e)}$, tj. vrijedi $(A^*)_{(e,f)}=(A_{(f,e)})^*$.

Ponovimo: $\left.\begin{array}{l}V \text{ unit.pr., } \langle\cdot|\cdot\rangle_V\\ W \text{ unit.pr., } \langle\cdot|\cdot\rangle_W\end{array}\right]\to$ $\forall A\in\mathcal{L}(V,W)$ $\exists!\, A^*\in\mathcal{L}(W,V)$ herm.adj. op. t.d. $\langle Av|w\rangle_W=\langle v|A^*w\rangle_V$ $\forall v\in V$ $\forall w\in W$

$\to$ vidimo da imamo preslikavanje: $\mathcal{L}(V,W)\overset{\text{v.p. iste dim; } \dim V=\dim W}{\to}\mathcal{L}(W,V)$, $A\mapsto A^*$ koje zadovoljava:

a) $(A^*)^*=A$ (involutornost)

b) $(\alpha A+\beta B)^*=\overline{\alpha}A^*+\overline{\beta}B^*$ (antilin.)

c) $V=W\to I\in\mathcal{L}(V,V)\wedge I=I^*$

9. UNITARNI I HERMITSKI OPERATORI

- to su pojmovi koji ovise o izboru sk. pr. $\langle \cdot | \cdot \rangle$ na V

Def. Operator $U \in \mathcal{L}(V)$ je **UNITARAN OPERATOR** ako vrijedi $\langle Ux | Uy \rangle = \langle x | y \rangle \quad \forall x, y \in V$
(tj. ako "čuva sk. pr.")

Prop. 1. $U \in \mathcal{L}(V)$ unitaran akko $UU^* = U^*U = I$.

DOKAZ:

$\Rightarrow$ Pretpostavimo da je U unitaran, tj. $\langle Ux | Uy \rangle = \langle x | y \rangle \quad \forall x, y \in V$

$\langle x | y \rangle = \langle Ux | Uy \rangle =$ (svojstvo herm. adj.) $= \langle x | U^*(Uy) \rangle \quad \forall x, y \in V$

$\to \langle x | y - U^*(Uy) \rangle = 0 \quad \forall x, y \in V$, sada uzmemo $x = y - U^*Uy$

$\to y - U^*(Uy) = 0 \to y = U^*Uy \to I = U^*U \quad \forall y \in V$

$\to U$ invert. ; $U^{-1} = U^* \to \boxed{UU^* = U^*U = I}$

$\Leftarrow$ Pretpostavimo $UU^* = U^*U = I$

$\langle Ux | Uy \rangle = \langle x | U^*(Uy) \rangle = \langle x | (U^*U)y \rangle = \langle x | Iy \rangle = \langle x | y \rangle \to \boxed{U \text{ unitaran}}$

Korolar 1. $\mathrm{Ker}(U) = \{0\}$.

DOKAZ:

1. način: U invertibilan $\to \boxed{\mathrm{Ker}(U) = \{0\}}$

2. način: $x \in \mathrm{Ker}(U) \to Ux = 0$

$\langle x | x \rangle = \langle Ux | Ux \rangle = \langle 0 | 0 \rangle = 0 \to x = 0 \to \boxed{\mathrm{Ker}(U) = \{0\}}$

Korolar 2. $W \subseteq V$ U-inv. ptp. $\to \forall y \in W \ \exists x \in W$ t.d. $Ux = y$

DOKAZ:

Ponovimo W inv. ptp ako $\forall w \in W$ vrijedi $Uw \in W$ te zato $U|W \in \mathcal{L}(W)$

Što je jezgra tom operatoru? $\mathrm{Ker}(U|W) = \{0\}$ (jer to vrijedi za $U \in \mathcal{L}(V)$; Kor. 1.)

Tm. o rangu i defektu: $\dim \mathrm{Im}(U|W) + \underbrace{\dim \mathrm{Ker}(U|W)}_{0} = \dim W \to \mathrm{Im}(U|_W) = W$ tj. $U|W$ surj.

$\forall y \in W \ \exists x \in W$ t.d. $Ux = y$

Prop. 2. Ako je W U-inv., onda je i njegov ort. kompl. $W^\perp$ također U-inv.
BITNO!

DOKAZ:

$W^\perp := \{x \in V ; \langle x | y \rangle = 0 \ \forall y \in W\}$ i to je lin. ptp. DZ i vrijedi $V = W \oplus W^\perp$ ort. suma (direktna, $W \perp W^\perp$)

Treba pokazati: $\forall y \in W^\perp \ Uy \in W^\perp$. $Uy \in W^\perp \leftrightarrow \langle Uy | z \rangle = 0 \quad \forall z \in W$

$\langle Uy | z \rangle \underset{\substack{W \text{ inv.} \\ z_1 \in W}}{=} \langle Uy | Uz_1 \rangle = \langle \underset{\in W^\perp}{y} | \underset{\in W}{z_1} \rangle = 0 \to Uy \in W^\perp \to \boxed{W^\perp \ U\text{-inv.}}$ (Kor. 2)

Prop. 3. Neka je $(e) = (e_1, \ldots, e_n)$ bilo koja ONB za V. Operator $U \in \mathcal{L}(V)$ je unitaran akko je matrica operatora U u bazi (e) unitarna matrica. (tj. $T = U_{(e)}$, U unit. $\leftrightarrow TT^* = T^*T = I$)
(Ako je $\mathbb{K} = \mathbb{R}$, $TT^* = T^*T = I$ govorimo o ortogonalnim matricama)

DOKAZ: Slijedi direktno iz Prop. 1.

Npr.
- $\mathbb{K} = \mathbb{C}$, $\alpha, \beta \in \mathbb{C}$, $|\alpha| = |\beta| = 1 \to T = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$ unitarna matrica
  Zaista, računamo: $T^* = \begin{bmatrix} \bar\alpha & 0 \\ 0 & \bar\beta \end{bmatrix}$, te zato $TT^* = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}\begin{bmatrix} \bar\alpha & 0 \\ 0 & \bar\beta \end{bmatrix} = \begin{bmatrix} \alpha\bar\alpha & 0 \\ 0 & \beta\bar\beta \end{bmatrix}$ $\begin{matrix} \alpha\bar\alpha = 1 = |\alpha|^2 \\ \beta\bar\beta = 1 = |\beta|^2 \end{matrix} \to TT^* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
- $\mathbb{K} = \mathbb{R}$, $\varphi \in [0, 2\pi)$, rot. oko ish. u smjeru suprotnom od kaz. na satu dana je matricom: $T = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$

  Tvrdimo da je T unitarna (kaže se ortogonalna)
  Računamo $TT^* = TT^t = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}\begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix} = \begin{bmatrix} \cos^2\varphi + \sin^2\varphi & \cos\varphi\sin\varphi - \sin\varphi\cos\varphi \\ \sin\varphi\cos\varphi - \cos\varphi\sin\varphi & \sin^2\varphi + \cos^2\varphi \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Prop. 4. Spektar unitarnog operatora sadržan je u jediničnoj kružnici. ($z \in \mathbb{C}, |z| = 1$) $\to S^1$
Spektar je uvijek neprazan ako je $\mathbb{K} = \mathbb{C}$ (kompl. unit. pr.)
Ako je $\mathbb{K} = \mathbb{R}$ (realni unit. pr.) spektar je sadržan u $\{\pm 1\}$ (može biti prazan)

DOKAZ:

Spektar operatora $U \in \mathcal{L}(V)$ je skup $\sigma_U = \{ \overset{\text{ključno!}}{\lambda \in \mathbb{K}} ; \lambda I - U$ nije $\overset{\overset{k_U(\lambda) = 0}{\leftrightarrow}}{\text{invert.}} \}$

$\left.\begin{matrix} \mathbb{K} = \mathbb{C} \to \sigma_U = \{\lambda \in \mathbb{C} ; k_U(\lambda) = 0\} \\ \mathbb{K} = \mathbb{R} \to \sigma_U = \{\lambda \in \mathbb{R} ; k_U(\lambda) = 0\} \end{matrix}\right] \to k_U(\lambda) = 0 \leftrightarrow \lambda I - U$ nije invert. $\to \mathrm{Ker}(\lambda I - U) \neq \{0_V\}$.

Izaberemo $0 \neq x \in \mathrm{Ker}(\lambda I - U) \to (\lambda I - U)x = 0 \to \lambda I x - Ux = 0 \to Ux = \lambda x \to$ x sv vekt. za sv. vrij. λ

$\left.\begin{matrix} U \text{ unitaran} \to \langle Ux | Ux \rangle = \langle x | x \rangle \\ \langle Ux | Ux \rangle = \langle \lambda x | \lambda x \rangle = |\lambda|^2 \langle x | x \rangle \end{matrix}\right] \to |\lambda|^2 = 1 \to \lambda = \pm 1 \to \boxed{\lambda \in S^1}$

$\mathbb{K} = \mathbb{C} \to \boxed{\sigma_U \neq \emptyset}$ (znamo!)

$\mathbb{K} = \mathbb{R} \to$ zbog def. spektra je $\lambda \in \mathbb{R} \xrightarrow{|\lambda| = 1} \lambda \in \{\pm 1\}$ tj. $\boxed{\sigma_U \subseteq \{\pm 1\}}$

- fali tm. o dijagonalizaciji unit op.

V unit. pr., sk. pr. $\langle\cdot|\cdot\rangle$

Def. $H \in \mathcal{L}(V)$ je HERMITSKI ako $\langle Hx|y\rangle = \langle x|Hy\rangle \ \forall x,y \in V$.

Ponovimo: Hermitsko adjungiranje; $A \in \mathcal{L}(V)$ tada $\exists A^* \in \mathcal{L}(V)$ t.d. $\langle Ax|y\rangle = \langle x|A^*y\rangle \ \forall x,y \in V$. Iz tog slijedi prop. 1.:

Prop 1. $H \in \mathcal{L}(V)$ hermitski akko $H = H^*$ (u terminima matrica)

Prop 2. Neka je $(e) = (e_1, \dots, e_n)$ bilo koja ONB za V. Znamo $H^*_{(e)}$ = herm. adj. od $H_{(e)} = H_{(e)}^*$
H hermitski akko $H_{(e)}^* = H_{(e)}$, tj. $H_{(e)}$ je hermitska matrica. ($T = t_{ij} \in M_{nn}(K)$ je herm. $\leftrightarrow t_{ij} = \overline{t_{ji}} \ \forall_{i,j}$)
T^* = transponiranje + konjug. svakog el. od T

Iz Prop. 2. imamo konstrukciju hermitskih operatora:
$K = R$ ili C, $T \in M_{nn}(K)$ herm. matrica. Pogledamo konkretan $V = M_{n\times 1}(K) = \left\{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}; x_i \in K\right\}$ na kojem imamo sk. pr.:
$\left\langle \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \middle| \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \right\rangle := \sum_{i=1}^{n} x_i \overline{y_i} = [x_1, \dots, x_n] \cdot \begin{bmatrix} \overline{y_1} \\ \vdots \\ \overline{y_n} \end{bmatrix} = x^t \overline{y}$
kanonska ONB: $e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \dots, e_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$ → sada definiramo lin. op.: $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \longmapsto \underbrace{Tx}_{\text{umn. matr.}} = \begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ t_{21} & t_{22} & \cdots & t_{2n} \\ \vdots & \vdots & & \vdots \\ t_{n1} & t_{n2} & \cdots & t_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

Pokazujemo da je hermitski (računamo s matricama):
$\langle Tx|y\rangle = (Tx)^t \cdot \overline{y} = x^t T^t \overline{y} = x^t \overline{Ty} = \langle x|Ty\rangle$, kako je $T = T^*$, tj. $t_{ij} = \overline{t_{ji}} \ \forall_{i,j}$, tj. $T^t = \overline{T}$, $\overline{T} := (\overline{t_{ij}})$

Korolar Ako je $K = R$, uz oznake iz Prop. 2., H hermitski akko $H_{(e)}$ sim. matrica, tj. $H_{(e)} = H_{(e)}^t$

Tm. 1. Neka je V u.p. (nad R ili C), sk. pr. $\langle\cdot|\cdot\rangle$. Tada: 1) $\sigma_H := \{\lambda \in K;\ \lambda I - H \text{ nije reg.}\} = \{\lambda \in K;\ k_H(\lambda) = \det(\lambda I - H) = 0\}$, tj. $\sigma_H \neq \emptyset$
2) $\sigma_H \subseteq R$

DOKAZ:

USM.! BITNO!

2) $\lambda \in \sigma_H \to \lambda I - H$ nije reg $\to \overbrace{\mathrm{Ker}\,(\lambda I - H)}^{\text{sv. ptp. za } \lambda} \neq \{0\}$. Uzmimo $x \in \mathrm{Ker}\,(\lambda I - H) \setminus \{0\}$ (x je sv. vektor)
H hermitski $\longrightarrow \langle Hx|x\rangle = \langle x|Hx\rangle$
$(\lambda I - H)x = 0 \to Hx = \lambda I x = \lambda x$
$\Big] \to \langle \lambda x|x\rangle = \langle x|\lambda x\rangle = \overline{\lambda}\langle x|x\rangle$, $\langle \lambda x|x\rangle = \lambda\langle x|x\rangle$ $\Big] \to \lambda = \overline{\lambda} \to \lambda \in R \to \underline{\sigma_H \subseteq R}$

1) $K = C$ (kompl. unit. pr.) (Dokaz za $K = R$ je drugačiji)
Znamo iz §2 $\sigma_H \neq \emptyset$, što slijedi iz osn. tm. algebre
fiksiramo ONB $(f) = (f_1, \dots, f_m)$ za V. Tada $T := H_{(f)}$ simetrična matrica $\in M_{n\times n}(R)$ po def kar. polinoma od $H \to k_A(\mu) = \det(\mu I_n - T) \ \forall \mu \in R$ te je dovoljno pokazati da postoji $\lambda \in R$ t.d. $\det(\lambda I_n - T) = 0$
Na ovom mjestu na trenutak zaboravimo sve prethodno osim da je <u>dana</u> simetrična matrica $T \in M_{nn}(R)$ $(T = (t_{ij}))$, $T = T^t$, tj. $t_{ij} = t_{ji} \ \forall i,j$; $t_{ij} \in R \to t_{ij} = \overline{t_{ji}} \ \forall_{ij}$ → T gledano kao matrica iz $M_{nn}(C)$ hermitska
što smo dobili? Pomoću T konstruiramo hermitski operator, pazite, na kompl. unit. pr
Traženi λ je bilo koja sv. vr. (kompl.) herm. operatora. Ona mora biti realna (vidi početak od (1) i (2))

Slično, kao kod unit. op., vrijedi:

Prop 3. $H \in \mathcal{L}(V)$ herm. op., $W \subseteq V$ H-inv. ptp. Tada: 1) $W^\perp$ H-invar. (ort. kompl. od W)
2) $H|W \in \mathcal{L}(W)$ i $H|W^\perp \in \mathcal{L}(W^\perp)$ su herm. op. na v.p. W i $W^\perp$

DOKAZ.

2) W H-invar. → $\forall x \in W \to Hx \in W$ te zato možemo def. $H|W \in \mathcal{L}(W)$ kao $(H|W)x = Hx \ \forall x \in W$
$\langle (H|W)x|y\rangle = \langle Hx|y\rangle \overset{H\text{ herm.}}{=} \langle x|Hy\rangle = \langle x|(H|W)y\rangle \ \forall x,y \in W \to H|W$ hermitski op. → 2)

1) $W^\perp = \{y \in V;\ \langle y|x\rangle = 0 \ \forall x \in W\}$, $V = W \oplus W^\perp$, $y \in W^\perp \overset{\text{tvrd.}}{\Rightarrow} Hy \in W^\perp$
$\forall x \in W \to Hx \in W$ (W je H-invar.) $\to Hx \perp W^\perp$
$\Big] \to Hx \perp y \to \langle Hx|y\rangle = 0 = \langle x|Hy\rangle \ \forall x \in W \to Hy \perp W \to Hy \in W^\perp \to \boxed{1)}$

2) $H|W^\perp$ herm se pokazuje analogno kao da je $H|W$ herm.

Tm.2. $H\in\mathcal{L}(V)$ herm. op. na real./kompl. unit. pr.. Tada postoji ONB $(e)=(e_1,\dots,e_n)$ za V i postoje $\lambda_1,\dots,\lambda_n\in\mathbb{R}$ t.d. $He_i=\lambda_i e_i$ $1\le i\le n$. (ekv.: H se dijagonalizira, $H_{(e)}=\begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix}$). Nadalje, $\sigma_H=\{\lambda_1,\dots,\lambda_n\}$.

DOKAZ: indukcijom

$Tm.1.\to\sigma_H\neq\emptyset$, $\sigma_H\subseteq\mathbb{R}$. Uzmemo $\lambda_1\in\sigma_H$ i $e_1\in Ker(\lambda_1 I-H)\setminus\{0\}$, tj. $He_1=\lambda_1 e_1$ (argum. kao u Tm.1.)

(eventualno zamjenjujući $e_1\leftrightarrow\frac{e_1}{\|e_1\|}$, možemo uzeti $\|e_1\|=1$, tj. e_1 je normiran)

bazni ind. ako $\dim V=1$: $W:=[\{e_1\}]=\{\mu e_1;\ \mu\in\mathbb{R}\}$ H-invar. jer $x\in W\to x=\mu e_1$ za neki $\mu\to Hx=H\mu e_1=\mu He_1=\mu(\lambda_1 e_1)\in\underbrace{[\{e_1\}]}_{=W}$

$\to W^\perp$ H-invar., $H|_{W^\perp}$ je herm. (Prop. 3.)

Ind. pretp. da je ONB $(e_1,\dots,e_n)$ za $W^\perp$ $\lambda_1,\dots,\lambda_n\in\mathbb{R}$ t.d. $\underbrace{(H|_{W^\perp})e_i}_{He_i}=\lambda_i e_i$

Spojimo $e_1 \overset{\text{još}}{\underset{\text{ubacimo}}{\longrightarrow}} (e_1,\dots,e_n)$ ON jer $W\perp W^\perp$, tj. $e_1\perp W^\perp$, $e_1\perp e_1,\dots,e_n$

(DZ) Tvrdnja oko spektra: vidi §4 ili izračunaj: $k_H(\mu)=\det(\mu I_n-H_{(e)})=(\mu-\lambda_1)\cdots(\mu-\lambda_n)\to$ $\sigma_H=\{\lambda_1,\dots,\lambda_n\}$

Def. Hermitski operator H je: a) POZITIVAN ako $\langle Hx|x\rangle\geq 0$ $\forall x\in V$, pišemo $H\geq 0$

b) STROGO POZITIVAN ako je poz. i $(\langle Hx|x\rangle=0\to x=0)$, pišemo $H>0$

Nap. $\langle Hx|x\rangle\in\mathbb{R}$ te prethodna def. ima smisla

$\langle Hx|x\rangle=(H\text{ herm})=\langle x|Hx\rangle\overset{sk.pr.}{=}\overline{\langle Hx|x\rangle}\to\langle Hx|x\rangle\in\mathbb{R}$

(DZ) **Prop.4.** $H\in\mathcal{L}(V)$, (e) ONB, $H_{(e)}=\begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix}$. Vrijedi: a) $H\geq 0\leftrightarrow\lambda_1,\dots,\lambda_n\geq 0$

b) $H>0\leftrightarrow\lambda_1,\dots,\lambda_n>0$

DOKAZ: $(e)=(e_1,\dots,e_n)$ ONB za V t.d. $Ae_j=\lambda_j e_j$ $j=1,\dots,n$

Neka je $x=\alpha_1 e_1+\dots+\alpha_n e_n$. $\to\langle Ax|x\rangle=\lambda_1|\alpha_1|^2+\dots+\lambda_n|\alpha_n|^2$

$Ax=\alpha_1\lambda_1 e_1+\dots+\alpha_n\lambda_n e_n$

• $A\geq 0$ $(A>0)\to$ za $x=e_j$ $\lambda_j|\alpha_j|^2>(\geq)0\to\lambda_j>(\geq)0$

• za $\lambda_1>(\geq)\dots\geq(>)\lambda_n\to$ očito $H\geq(>)0$

(DZ) **Prop.5.** $A\in\mathcal{L}(V)$ (bilo koji operator). Tada: $H:=A^*A$ hermitski i $H\geq 0$.

Nadalje; $\det A\neq 0\to H>0$.

DOKAZ:

• $(A^*A)^*=A^*(A^*)^*=A^*A\to A^*A$ herm.

• $\langle A^*Ax|x\rangle=\langle Ax|Ax\rangle=\|Ax\|^2\geq 0\to H\geq 0$

• $\det A\neq 0\to A$ reg. $\to Ax\neq 0$ $\forall x\neq 0$ $\|Ax\|^2>$ $\forall x\neq 0\to A^*A>0$

(DZ) **Tm.** Neka je A pozitivan operator. Tada $\exists!\,B>0$ t.d. $B^2=A$.

B zovemo DRUGI KORIJEN IZ A; pišemo $B=\sqrt{A}$.

(DZ) **Tm.** (POLARNA FORMA OPERATORA)

$A\in\mathcal{L}(V)$. Tada $\exists U$ unit. op. i $\exists H>0$ op. t.d. $A=UH$.

H jest jedinstven, i $H=\sqrt{A^*A}$. A reg. $\to U$ jedinstven.

10. NORMALNI OPERATORI

V unit. realan ili V unit. kompl. v.p., sk.pr. $\langle\cdot|\cdot\rangle$

Def. Operator $N\in\mathcal{L}(V)$ je NORMALAN ako komutira sa svojim hem.adj.op. (u odnosu na zadani sk.pr.), tj. $NN^*=N^*N$

Prop.1. Svaki unitaran ili hermitski op. je normalan operator.

DOKAZ:

- U unit.op. $\to U^{-1}=U^*$ inverz tj. $UU^*=U^*U=I \to$ komutiraju $\to U$ normalan
- H herm.op. $\to H=H^* \to HH^*=HH=H^2=H\cdot H=H^*H \to$ komutiraju $\to H$ normalan

Primjer: • V kompl.unit.pr. $\to \sigma_{unit.op.}\subseteq S^1=\{z\in\mathbb{C};\ |z|=1\} \to U$ definiran matricom $U_{(ONB)}=\begin{bmatrix} e^{i\varphi} & 0\\ 0 & e^{i\psi}\end{bmatrix}$, $\varphi,\psi\in\mathbb{R}$ unit.

$\to \sigma_{HERM.op}\subseteq\mathbb{R} \to H$ definiran matricom $H_{(ONB)}=\begin{bmatrix}\lambda & 0\\ 0 & \mu\end{bmatrix}$, $\lambda,\mu\in\mathbb{R}$ herm.

$\overline{e^{i\varphi}}=$ $\cos\varphi-i\sin\varphi=$ $\cos(-\varphi)+i\sin(-\varphi)=$ $e^{-i\varphi}$

$$\begin{bmatrix} e^{i\varphi} & 0\\ 0 & e^{i\psi}\end{bmatrix}\begin{bmatrix} e^{i\varphi} & 0\\ 0 & e^{i\psi}\end{bmatrix}^*=\begin{bmatrix} e^{i\varphi} & 0\\ 0 & e^{i\psi}\end{bmatrix}\begin{bmatrix} \overline{e^{i\varphi}} & 0\\ 0 & \overline{e^{i\psi}}\end{bmatrix}=\begin{bmatrix} e^{i\varphi} & 0\\ 0 & e^{i\psi}\end{bmatrix}\begin{bmatrix} e^{-i\varphi} & 0\\ 0 & e^{-i\psi}\end{bmatrix}=\begin{bmatrix} 1 & 0\\ 0 & 1\end{bmatrix}\to U \text{ unit.}$$

$$\begin{bmatrix}\lambda & 0\\ 0 & \mu\end{bmatrix}^*=\begin{bmatrix}\bar\lambda & 0\\ 0 & \bar\mu\end{bmatrix}=\begin{bmatrix}\lambda & 0\\ 0 & \mu\end{bmatrix} \text{ (jer su } \lambda,\mu\in\mathbb{R}) \to H \text{ herm.}$$

• $\to N$ definiran matricom $N_{(ONB)}=\begin{bmatrix}\lambda & 0\\ 0 & \mu\end{bmatrix}$, $\lambda,\mu\in\mathbb{C}$ (bilo koji) nisu nužno $\lambda=e^{i\varphi}$, $\mu=e^{i\psi}$ tj. na S^1, niti $\lambda,\mu\in\mathbb{R}$

$\begin{bmatrix}\lambda & 0\\ 0 & \mu\end{bmatrix}^*=\begin{bmatrix}\bar\lambda & 0\\ 0 & \bar\mu\end{bmatrix}$ te zato $\begin{bmatrix}\lambda & 0\\ 0 & \mu\end{bmatrix}\begin{bmatrix}\lambda & 0\\ 0 & \mu\end{bmatrix}^*=\begin{bmatrix}\lambda & 0\\ 0 & \mu\end{bmatrix}\begin{bmatrix}\bar\lambda & 0\\ 0 & \bar\mu\end{bmatrix}=\begin{bmatrix}\lambda\bar\lambda & 0\\ 0 & \mu\bar\mu\end{bmatrix}$

$\begin{bmatrix}\lambda & 0\\ 0 & \mu\end{bmatrix}^*\begin{bmatrix}\lambda & 0\\ 0 & \mu\end{bmatrix}=\begin{bmatrix}\bar\lambda & 0\\ 0 & \bar\mu\end{bmatrix}\begin{bmatrix}\lambda & 0\\ 0 & \mu\end{bmatrix}=\begin{bmatrix}\lambda\bar\lambda & 0\\ 0 & \mu\bar\mu\end{bmatrix}$ $\to N$ normalan, ali nije nužno niti unit. niti herm.

Nap. Klasa normalnih op. je šira od unije klasa unit. i herm. op.

Prop.2. (KARAKTERIZACIJA NORM. OP.)

Neka je V u.v.p. ($\mathbb{R}$ ili $\mathbb{C}$). Operator $N\in\mathcal{L}(V)$ normalan akko $\|Nv\|=\|N^*v\|$, $\forall v\in V$.

DOKAZ

$\boxed{\Rightarrow}$ Pretp. da je $N\in\mathcal{L}(V)$ normalan, tj. $NN^*=N^*N$

Neka je $v\in V$. Računamo: $\|Nv\|^2=\langle Nv|Nv\rangle \overset{\text{svojstvo herm.adj.}}{=} \langle v|N^*(Nv)\rangle=\langle v|\underbrace{(N^*N)}_{NN^*}v\rangle=\underbrace{\langle v|N(N^*v)\rangle}_{(N^*)^*=N}$
$=\langle N^*v|N^*v\rangle=\|N^*v\|^2$

slučaj: $\boxed{\Leftarrow}$ (kompl.) Koristimo činjenicu da se herm. op. na realnom ili kompl. v.p. dijagonalizira (realan na dijagonali) u nekoj ONB
$H=H^*\to \exists$ ONB $(e)=(e_1,\dots,e_n)$ za V i $\exists\lambda_1,\dots,\lambda_n\in\mathbb{R}$ t.d. $H_{(e)}=\begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix}$
$He_i=\lambda_ie_i$ $1\le i\le n$ $\wedge$ (e) ONB za V) $\to H=0\Leftrightarrow\lambda_1=\dots=\lambda_n=0$

Pretp. da vrijedi: $\|Nv\|=\|N^*v\|\to\|Nv\|^2=\|N^*v\|^2\to\langle \underset{N^*}{N}v|Nv\rangle=\langle \underset{(N^*)^*=N}{N^*}v|N^*v\rangle$

$\to\langle N^*Nv|v\rangle=\langle NN^*v|v\rangle\to\langle N^*Nv-NN^*v|v\rangle=0\to\langle(N^*N-NN^*)v|v\rangle=0$ $\forall v\in V$

$(AB)^*=B^*A^*$

$H:=N^*N-NN^*$ taj op. je hermitski jer $H^*=(N^*N-NN^*)^*=(N^*N)^*-(NN^*)^*=N^*(N^*)^*-(N^*)^*N^*$
$=N^*N-NN^*=H$

$\langle Hv|v\rangle=0$ $\forall v\in V$ te ako uzmemo redom $v=e_1,\dots,e_n\to 0=\langle He_i|e_i\rangle=\langle\lambda_ie_i|e_i\rangle=\lambda_i\underbrace{\langle e_i|e_i\rangle}=\lambda_i$
$\to\lambda_1=\dots=\lambda_n=0\to H=0$ nul.op. $\to N^*N-NN^*=H=0\to N^*N=NN^*\to$ N norm.

Prop.3. $N\in\mathcal{L}(V)$ normalan $\wedge$ $\lambda\in\mathbb{K}\to N-\lambda I$ normalan

DOKAZ: (DZ)

Uputa: $(N-\lambda I)^*=N^*-\bar\lambda I$

$(N-\lambda I)(N-\lambda I)^*=(N-\lambda I)(N^*-\bar\lambda I)=$

$\|(N-\lambda I)v\|=0 \Leftrightarrow \|(N-\lambda I)^*v\|=\|(N^*-\bar\lambda I)v\|=\dots$

~~Prop.4.~~ $N\in\mathcal{L}(V)$ normalan op., $\lambda\in\sigma_N$. Tada: 1) $\bar\lambda\in\sigma_{N^*}$

2) $v\in\operatorname{Ker}(N-\lambda I)$ sv. vektor (tj. bilo koji $\neq 0$) $\to v\in\operatorname{Ker}(N^*-\bar\lambda I)$
(tj. $Nv=\lambda v\Rightarrow N^*=\bar\lambda v$)

3) $\operatorname{Ker}(N-\lambda I)=\operatorname{Ker}(N^*-\bar\lambda I)$

Prop.4!: $N\in\mathcal{L}(V)$ normalan op. $\lambda\in\sigma_N$, v sv.vektor za λ (tj. $v\neq 0$ t.d. $Nv=\lambda v$). Tada je: $\bar\lambda\in\sigma_{N^*}$ i v sv.vekt. za $\bar\lambda$.

DOKAZ

$N_\lambda:=N-\lambda I$ je normalan i vrijedi $N_\lambda^*=N^*-\bar\lambda I$ (Prop.3.), te zato (Prop.2.) pokazuje
$\|N_\lambda v\|=\|N_\lambda^*v\|$. Drugim riječima: $\|(N-\lambda I)v\|=\|(N^*-\bar\lambda I)v\|$

jer $Nv=\lambda v$, $Nv-\lambda v=0$ $\leftarrow 0=\|Nv-\lambda v\|=\|Nv-\lambda Iv\|$ analogno $\|N^*v-\bar\lambda v\|$ te mora biti: $N^*v-\bar\lambda v=0\to N^*v=\bar\lambda v$

Ponovimo:
$\mathrm{Ker}\, A = \{x \in V ; Ax = 0\}$
$\mathrm{Im}\, A = \{Ax ; x \in V\}$

Prop 5. Neka je $A \in \mathcal{L}(V)$ operator. Vrijedi $V = \mathrm{Ker}(A) \oplus \mathrm{Im}(A^*)$.
Ako je A još normalan onda $V = \mathrm{Ker}(A) \oplus \mathrm{Im}(A)$.

Dokaz:

- $V = \mathrm{Ker}(A) \oplus \mathrm{Im}(A^*)$ znači $[\mathrm{Im}(A^*)]^\perp = \mathrm{Ker}(A)$
  $x \in [\mathrm{Im}(A^*)]^\perp \Leftrightarrow x \perp \mathrm{Im}(A^*) \Leftrightarrow x \perp A^*y \ \forall y \in V \Leftrightarrow \langle x | A^*y \rangle = 0 \ \forall y \in V \Leftrightarrow \langle Ax | y \rangle = 0 \ \forall y \in V$
  $\mathrm{Im}(A^*) = \{A^*y ; y \in V\}$
  te posebno za $y = Ax \Leftrightarrow Ax = 0 \Leftrightarrow \underline{\underline{x \in \mathrm{Ker}\, A}}$
- za 2. tvrdnju koristimo Prop. 4., $\lambda = 0$. $\mathrm{Ker}\, A = \mathrm{Ker}(A - 0 \cdot I) = \mathrm{Ker}(A^* - \bar{0} I) = \mathrm{Ker}(A^*)$
  Sada koristimo 1. dio dokaza: za A^*: $V = \mathrm{Ker}(A^*) \oplus \mathrm{Im}(\underbrace{(A^*)^*}_{A}) = \mathrm{Ker}(A) \oplus \mathrm{Im}(A)$

TM. V kompl. unit. pr. i $N \in \mathcal{L}(V)$ normalan op. Tada postoji ONB $(e) = (e_1, \dots, e_n)$ za V, postoje $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ t.d. $N_{(e)} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$, dijagonalizira se. (Drugim riječima $Ne_i = \lambda_i e_i$, $1 \le i \le n$, $\sigma_N = \{\lambda_1, \dots, \lambda_n\}$

Dokaz (skica)

Indukcijom $\dim V$. korak

V nad $\mathbb{C} \to \sigma_N \neq \emptyset$. Uzmemo neki $\lambda_1 \in \sigma_N$.

e_1 normirani sv. vektor za λ_1

$Ne_1 = \lambda_1 e_1 \xrightarrow[\text{ključno prop 4.}]{} N^* e_1 = \bar{\lambda}_1 e_1$

$V = \langle e_1 \rangle \oplus \langle e_1 \rangle^\perp$ ključno i glavna stvar za indukcije

$\langle e_1 \rangle^\perp$ je N i N^*-inv. te zato $N|_{\langle e_1 \rangle^\perp}$ norm. op. čiji hermadj. op $N^*|_{\langle e_1 \rangle^\perp}$

(i sad dalje indukcijom)

<u>Treba</u> $\forall x \in \langle e_1 \rangle^\perp \to Nx, N^*x \in \langle e_1 \rangle^\perp$, tj. $Nx \perp e_1$, $N^*x \perp e_1$

$\langle Nx | e_1 \rangle \underset{\text{HERM ADJ}}{=} \langle x | N^* e_1 \rangle = \langle x | \bar{\lambda}_1 e_1 \rangle = \overset{\lambda_1}{\bar{\bar{\lambda}}_1} \underbrace{\langle x | e_1 \rangle}_{0} = 0$

$\langle N^* x | e_1 \rangle = \langle x | N e_1 \rangle = \langle x | \lambda_1 e_1 \rangle = \bar{\lambda}_1 \langle x | e_1 \rangle = 0$